Twisted Trace Formula for non-compactly supported test functions

1. INTRODUCTION

F: number field

G: Connected reductive gp over F

$\theta : G \to G$, F-autom. of finite order.

$\tilde{G} = G \rtimes \theta$ coset of $G \rtimes \langle \theta \rangle$.

Arthur-Selberg Trace Formula:

* Selberg (1956) $G = SL(2)$
* J. Arthur (late 70s) General G.
* Identity of distributions:

$$J^G_{geom}(f) = J^G_{spec}(f), \quad f \in C_c^\infty(G(\mathbb{A})).$$

* Geometric side: Weighted orbital integrals
* Spectral side: Sum of traces of reps in the discrete spectrum of G.

Twisted trace formula:

* Clozel-Labesse-Langlands (1983-84 FMS @ IAS)
* Trace formula for $\tilde{G}$, i.e.

$$J_{geom}^{\tilde{G}}(\cdot) = J_{spec}^{\tilde{G}}(\cdot), \qquad f \in C_c^{\infty}(\tilde{G}(\mathbb{A}))$$

* Trace over reps of G s.t. $\pi \sim \pi^{\theta}$.

Applications (of ttf) to Functoriality:

(1) Cyclic base change (Arthur-Clozel)

$G = GL(n)$

E/F : cyclic extn. of number fields

$\langle \theta \rangle = Gal(E/F)$ &

$\theta : GL(n, \mathbb{A}_E) \longrightarrow GL(n, \mathbb{A}_E)$

Autom. reps of $GL(n, \mathbb{A}_F)$ $\longleftrightarrow$ θ-invariant autom. reps of $GL(n, \mathbb{A}_E)$

(2) Classical groups (~~✗~~ Arthur)

$G = GL(N)$

$\theta(x) = {}^t x^{-1}$

Autom. reps of $SO(2n), Sp(2n), SO(2n+1)$ + Local Langlands $\longleftrightarrow$ selfdual autom. reps of $GL(N)$

(3) Unitary gps (Mok)

$\theta(x) = {}^t \bar{x}^{-1}$.

The proof in these cases goes by comparing trace formulae.

Arthur has shown that terms in the geometric expansion are finite:

$$J^G_{geom}(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J^G_{\mathfrak{o}}(f)$$

and $J^G_{\mathfrak{o}}(f) \leq \int |k_{\mathfrak{o}}(x)|dx < \infty$,

but not a similar result on the spectral side.

Rather in his comparison of trace formulae, he didn't <u>need</u> the absolute convergence of the spectral expansion, which was proven by

* Müller-Speh (2004) for $G=GL(N)$
* Finis-Lapid-Müller (2011) for general G.

FLM proved the result for a broader class of test functions.

Fix $\underline{K} = K_\infty K_f$, good maximal compact subgp of $G(\mathbb{A})$

& choose $K \leq K_f$ open compact subgp.

$$C(G(\mathbb{A}),K) = \left\{ f : G(\mathbb{A}) \longrightarrow \mathbb{C} \text{ smooth \& right } K\text{-invariant, } \|f * X\|_{L^1} < \infty \ \forall X \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \right\}$$

$\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G) \otimes \mathbb{C}$.

X: invariant differential operators acting on Archimedean component of f.

<u>Thm (FLM)</u> 2011 $J^G_{spec}(f)$ converges absolutely for $f \in C(G(\mathbb{A}),K)$.

<u>Thm (FL, 2015)</u> The geometric side $J^G_{geom}(f)$ is valid for $f \in C(G(\mathbb{A}),K)$.

Thus we have a broader trace formula. The main result of this talk is the generalization to the twisted case.

2. SPECTRAL SIDE

$$J^{\tilde{G}}_{spec}(f) = \overbrace{\sum_{\tilde{L} \in \mathcal{L}^{\tilde{G}}} \sum_{M \in \mathcal{L}^{L}} \sum_{\tilde{w} \in W^{\tilde{L}}(M)_{reg}}}^{\text{finite sums}} [\cdot] \int_{i(\mathfrak{a}^{\tilde{G}}_{\tilde{L}})^*}$$

$$\operatorname{trace}\left(\mathcal{M}^{\tilde{G}}_{\tilde{L}}(P,\nu)\, M_{P|\tilde{w}P}(0)\, \rho_{P,disc,\nu}(\tilde{w},f)\right) d\nu$$

<u>Theorem</u> (P.) The integral above is absolutely convergent wrt. trace norm and extends to ~~$f \in \mathcal{C}$~~ $f \in \mathcal{C}(\tilde{G}(\mathbb{A}), K)$ (defined analogously). Thus $J^{\tilde{G}}_{spec}(\cdot)$ is a continuous distribution.

(Will give an application if time permits).

<u>Proof idea</u>

$\mathcal{M}^{\tilde{G}}_{\tilde{L}}(P,\nu)$ is the smooth ~~oper~~ function corresponding to the $(\tilde{G},\tilde{M})$-family of intertwining operators. [FL] have given a formula for this operator as a finite sum of derivatives of intertwining operators. This operator is the restriction of the non-twisted operator on some θ-invariant space. So the estimates of FL, 2011 ~~lemma~~ hold.

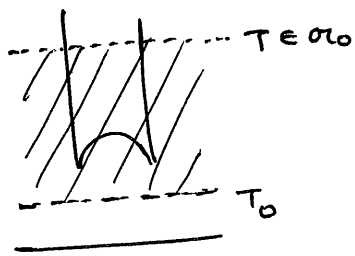
More notation:

A_G: Connected component of the $\mathbb{R}$-points of the max. split torus in the center of G.

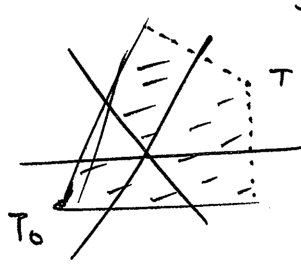
$X(G)_F$ = F-rational characters of G.

$\mathfrak{a}_G = \mathrm{Hom}(X(G)_F, \mathbb{R})$.

$X_G = A_G G(F) \backslash G(\mathbb{A})$; For $T \in \mathfrak{a}_0$, X_G^T: approximate Siegel set.



SL(2)



SL(3)

$\mathcal{A}(X_P, \sigma) \subseteq \mathcal{A}(X_P)$: space of automorphic forms.

$$\rho_{P,\mathrm{disc},\nu}(\tilde{w}, f) : \mathcal{A}(X_P) \longrightarrow \mathcal{A}(X_{\tilde{w}P})$$

is the operator

$$\rho_{P,\mathrm{disc},\nu}(\tilde{w}, f) = \int_{y \in \tilde{G}(\mathbb{A})} f(y) \underbrace{\rho_{P,\mathrm{disc},\nu}(\tilde{w}, y)}_{\text{“}[\Phi(x) \mapsto \Phi(\tilde{w}^{-1}xy)]\text{”}} dy$$

And $\quad M_{P|\tilde{w}P} : \mathcal{A}(X_{\tilde{w}P}) \longrightarrow \mathcal{A}(X_P)$

is the usual intertwining operator, so the composition

$M_{P|\tilde{w}P} \circ \rho_{P,\mathrm{disc},\nu}(\tilde{w}, f)$ is an operator on $\mathcal{A}(X_P)$.

We introduce a shift operator which is unitary, to get to the non-twisted setting. $\square$

3. GEOMETRIC SIDE

Assume the Root Cone Lemma

* proved in "all" interesting cases
* But not in general.

Roughly the geometric expansion is,

$$J^T(f) = \int_{X_G} \cancel{k^T_f}\ k^T(x)\,dx, \qquad T \in \mathfrak{a}_0,$$

where
$$k^T(x) = \sum_{P \supseteq P_0} (-1)^{a_P - a_G} \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} k_P(\gamma x)\, \hat{\tau}_P(H_P(\gamma x) - T)$$

$$k_P(\gamma x) \downarrow \quad \sum_{\delta \in M_P(\mathbb{Q})} \int_{N_P(\mathbb{A})} f(x^{-1}\delta n x)\,dn.$$

And the geometric side can be decomposed ~~into~~ according to (coarse) conjugacy classes as

$$k^T(x) = \sum_{\mathfrak{o} \in \mathcal{O}} k^T_{\mathfrak{o}}(x).$$

Arthur showed ~~this~~ the distributions $J^T(f)$, $J^T_{\mathfrak{o}}(f)$ are polynomials in T and

$$J(f) = J_{geom}(f) := J^{T_0}(f),$$

$T_0 \in \mathfrak{a}_0$ is a special point.

Then,
$$J_{geom}(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f).$$

coarse conjugacy classes.

<u>Theorem</u> (P.) Fix $f \in C(\tilde{G}(\mathbb{A}), K)$.

(1) $J^T(f) := \int k^T(x)\,dx$ and $J^T_{\mathfrak{o}}(f) = \int \cancel{k^T_{\mathfrak{o}}(f)}\; k^T_{\mathfrak{o}}(x)\,dx$

are absolutely convergent for "T large".

(2) Polynomials in T. ~~of~~

(3) $$\sum_{\mathfrak{o} \in \mathcal{O}} \left| J^T_{\mathfrak{o}}(f) - \int k^T_{\mathfrak{o}}(x)\,dx \right| \leq \mu(f)\,(1+\|T\|)^r\, e^{-d_0(T)}$$

μ seminorm on $C(\tilde{G}(\mathbb{A}), K)$, $r > 0$.

(4) $J^T_{geom}(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J^T_{\mathfrak{o}}(f)$ holds.

<u>Remark</u> The bound on μ is known in terms of $K \leq K_f$.

(For application to limit multiplicity).

4. ROOT CONE LEMMA

The convergence on the geometric side is conditional on the following lemma.

Lemma Let $w \in W$, $w \neq 1$. There exists an open cone Ω in the positive Weyl chamber $(\mathfrak{a}_0^*)^+$ such that if $\lambda \in \Omega$,

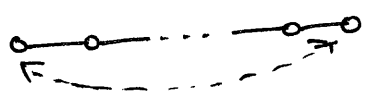
$$\langle \lambda - \theta^{-1} w^{-1} \lambda , \varpi_\beta^\vee \rangle > 0 \qquad \forall \beta \in \Delta_0^{Q(w)} \subseteq \Delta_0 .$$

Remark In the usual (non-twisted) setting, $\Omega = (\mathfrak{a}_0^*)^+$ works.

Proven RCL in these cases :-

(0) Reduction to G simple

(1) $G = GL(n)$ or type A_n, $\qquad \theta(x) = {}^t x^{-1}$

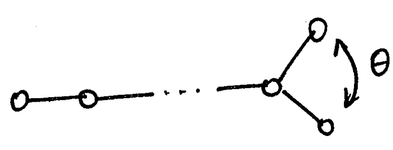


(2) Base change :

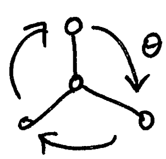
$$\mathrm{Gal}(E/F) = \langle \theta \rangle$$

$$G = \mathrm{Res}_{E/F} H_E , \qquad \theta : G(E) \to G(E).$$

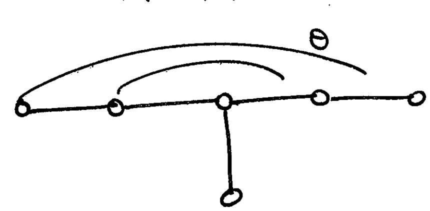
(3) G of type D_n



type D_4



4. Type E_6, using SAGEMATH



Questions

(*) ~~Gen~~ G not split?

(*) Proof without case-by-case exhaustion.

5. APPLICATION

E/F number fields

$\Gamma = \mathrm{Gal}(E/F) = \langle \sigma, \theta \rangle$.

$G = GL(n)_E$, $K = \prod_w K_w$ usual max. compact subgp of $GL(n, \mathbb{A}_E)$.

Nonsolvable base change:

Γ-invariant autom. reps of $GL(n, \mathbb{A}_E)$ $\longleftrightarrow$ Autom. reps of $GL(n, \mathbb{A}_F)$.

Fix $S \supseteq S_\infty$ finite set of places of E.

For $v \in S$, pick φ_v: smooth right K_v-invariant fn. on $G(E_v)$.

If $v = \infty$, assume $\|\varphi_v * X\| < \infty$ $\forall X \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.

Let $\varphi_S = \prod \varphi_v$.

Then

$$\sum_{\substack{\pi \text{ cuspidal} \\ \text{autom. rep} \\ \text{of } GL(n, \mathbb{A}_E) \\ \pi \sim \pi^\Gamma}} \mathrm{tr}\, \pi_S(\varphi_S) \cdot \operatorname*{Res}_{s=1} L^S(s, \pi \times \pi^{\vee\sigma}) < \infty$$

$\downarrow$

Rankin-Selberg L-function (simple pole if $\pi \sim \pi^\sigma$)

<u>Proof</u> Uses Langlands' basic functions ~~on $J^{\tilde{G}}_{spec}(f^0)$~~. to $|J^{\tilde{G}}_{spec}(\cdot)| < \infty$. $\square$