

- Thank you for all your help & teaching
- Wish you a very happy birthday & many more years of beautiful mathematics.

① Thank you for your help

KHEILI MAMNOON AZ KOMAK-E-SHOMA.

② Wish you a happy birthday

TAKALOD-E-SHOMA MOBARRAK

③ wish you many more years of beautiful mathematics

~~SAD SAAL-HAYE~~

SAAL-HAYE ZIAD-E RIAZI-E-ZIBA

BARAYETAN AREZOO DARAM.

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ZAAJ-HAVE ZIAD-E KHASIFZIBA

BARAYETAN AKESD DARAN

Eisenstein Cohomology & Special values of L-fun.

(1)

§1 Classical results:

Theorem (Mazur, Shimura)

$\varphi \in S_k(\Gamma_0(N), \omega)_{\text{prim}}$, $\exists u^\pm(\varphi) \in \mathbb{C}^\times$ Periods
 \forall Dirichlet char. χ , $1 \leq m \leq k-1$ critical set for $L(s, \varphi, \chi)$
 $L_f(m, \varphi, \chi) \approx (2\pi i)^m \cdot u^\pm(\varphi) \cdot g(\chi)$; $\pm(-1)^m = \chi(-1)$

Cor: $1 \leq m \leq k-2$, $\frac{L_f(m, \varphi, \chi)}{L_f(m+1, \varphi, \chi)} \approx \frac{1}{(2\pi i)} \cdot \frac{u^\pm(\varphi)}{u^\mp(\varphi)}$

Cor: $\Omega(\varphi) = \frac{1}{i} \frac{\int u^\pm(\varphi)}{u^\mp(\varphi)}$

$\frac{L(m, \varphi, \chi)}{L(m+1, \varphi, \chi)} \approx \Omega(\varphi)^{\sum \epsilon_i \epsilon_i}$

$GL_n \times GL_{n'}/F$
 n -even, n' -odd

Similar Theorems for Hilbert mod. forms. $GL_2 \times GL_1/F$

Theorem (Shimura; Math. Ann. 77)

$\varphi \in S_k(\Gamma_0(N), \omega)$, $\xi \in S_l(\Gamma_0(N), \psi)$, Assume $k > l$.
 $l \leq m < k$ critical set.
 $L_f(m, \varphi \times \xi) \approx (2\pi i)^{2m+1-l} \cdot u^+(\varphi) u^-(\psi) \cdot g(\psi)$

Cor: $\frac{L_f(m, \varphi \times \xi)}{L_f(m+1, \varphi \times \xi)} \approx \frac{1}{(2\pi i)^2}$

$L(m, \varphi \times \xi) \approx L(m+1, \varphi \times \xi)$

Similar Thm. for Hilbert mod. forms.

$GL_2 \times GL_2$

$GL_n \times GL_{n'}$, n, n' both even.

Harder + R

$SO(n, n) \times GL_1$

(Blagovest + R)

§2 Scope of the theorems on special values of L-fun. (via Eisenstein coh.)

Theorem (1) (Harder + R)

σ - cusp. ant. repⁿ. of $GL_n(\mathbb{A}_F)$, F -tot. real, $\sigma \in \text{Coh}(GL_n/F, \mu)$

σ' - " " " " " $GL_{n'}$, $\sigma' \in \text{Coh}(GL_{n'}/F, \mu')$, $N = n + n'$

Assume a certain condition $\mathcal{E}(\mu, \mu')$ on μ & μ' .

n -even, n' -odd $\frac{L(-N/2, \sigma \times \sigma')}{L(1-N/2, \sigma \times \sigma')} \approx \Omega(\sigma)^{\varepsilon_{\sigma'}}$

Can replace σ by $\sigma \otimes \|\cdot\|^n$, $n \in \mathbb{Z}$ as long as $\mathcal{E}(\mu \otimes \|\cdot\|^n, \mu')$ is satisfied

~~if~~ if m & $m+1$ are critical then

$$\frac{L(m, \sigma \times \sigma')}{L(m+1, \sigma \times \sigma')} \approx \Omega(\sigma)^{\varepsilon_{\sigma'} \varepsilon_m}$$

n, n' both even,

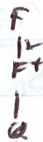
$$L(m, \sigma \times \sigma') \approx L(m+1, \sigma \times \sigma')$$

Theorem (2) (R)

σ, σ' as above, F -CM field

$\mathcal{E}(\mu, \mu')$ holds.

$m, m+1$ critical for $L(1, \sigma \times \sigma')$



$$L(m, \sigma \times \sigma') \approx L(m+1, \sigma \times \sigma')$$

Theorem (3) (Bhargava + R)

σ - cusp. ant. repⁿ. of $SO(n, n)/F_{\neq \mathbb{Q}}$, F -tot. real, $\sigma \in \text{Coh}(SO(n, n)/F, \mu)$

Assume n -even.

σ - cusp.

σ - globally generic.

$\sigma \otimes \chi$ - d.n. repⁿ. H.C. param $\mu + \rho$. $\mu = \mu_1 \geq \dots \geq \mu_{n/2} \geq |\mu_{n/2}|$

χ - fin. order char of $\mathbb{A}_F^\times / F^\times \rightarrow \mathbb{C}^\times$

if $m, m+1 \in \{1 - |\mu_{n/2}|, \dots, |\mu_{n/2} - 1|, |\mu_{n/2}|\}$ then

$$L(m, \sigma \times \chi) \approx L(m+1, \sigma \times \chi)$$

" $k \geq l$ "

Assume

$|\mu_{n/2}| \geq 1$

My student Charitangya Anubi $SO(n, n)$ / CM-field

Ultimate goal: Study arithmetic of Langlands-Shahidi L-fun.

§3 Analytic Theory of certain L-fns.

G-comm. red. grp / Q.

P - non. parabolic $P = M_P N_P$, α - simple root deleted, $\tilde{\alpha}$ - fund. wt.

σ - arb. ant. repⁿ of $M(A)$ (assume σ -generic / not needed for many steps)

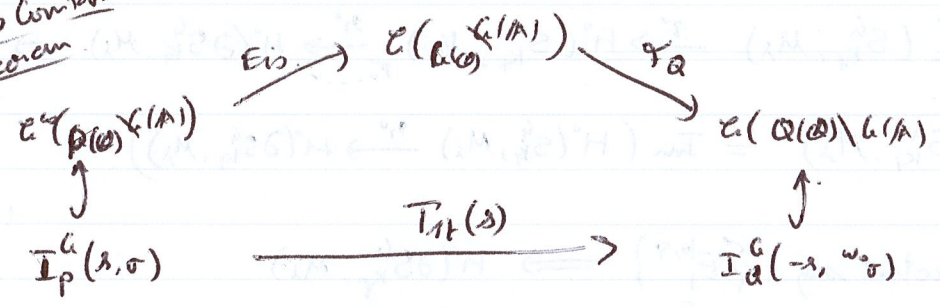
$$f \in I_P^G(\tilde{\alpha}, \sigma) = \text{Inv}_{P(A)}^{G(A)}(\sigma \otimes \tilde{\alpha}) \hookrightarrow C_c^\infty(P(A) \backslash G(A))$$

$$\text{Eis}_P(\lambda, f)(g) = \sum_{\gamma \in P(A) \backslash G(A)} f(\gamma g) \in C_c^\infty(G(A))$$

Q - associate parabolic.

$$\Psi_Q : C_c^\infty(Q(A) \backslash G(A)) \rightarrow C_c^\infty(Q(A) \backslash G(A)) \quad \Psi_Q(\varphi)(g) = \int_{N_Q(A)} \varphi(ng) dg$$

Langlands Constant Term Theorem



$$T_{1st} = \bigotimes_v T_{1st,v}, \quad \forall \lambda, \quad T_{1st,v}(f_v) = \left(\prod_{i=1}^m \frac{L(\lambda, \sigma_v, \tilde{\chi}_i)}{L(\lambda+1, \sigma_v, \tilde{\chi}_i)} \right) \cdot \tilde{E}_{f_v}^0$$

$G = GL_n$
 $P = Perm$

$T_{1st}(\lambda)$ will see, up to $v \in S_{\text{un}}$, $v \in S_{\text{ram}}$, $\frac{L^S(\lambda, \sigma \times \sigma^{-v})}{L^S(\lambda+1, \sigma \times \sigma^{-v})}$

~~Eisenstein cohomology is designed to see the constant term thm in cohomology.~~

Eisenstein cohomology gives a cohomological interpretation to the constant term theorem.

S4 Eisenstein Cohomology $G/\mathbb{Q} \supset B/\mathbb{Q} \supset T/\mathbb{Q} \supset Z/\mathbb{Q} \supset S/\mathbb{Q}$

$\lambda \in X_+^*(T)$

$M_{\lambda, E}$, alg. irred. repⁿ. of $G \times E$ with h.w. λ .

Cos - max. split in $G(\mathbb{R})$, $K_{cos} = Cos \cdot S(\mathbb{R})$ $K_{cos}^0 = \text{conn. comp. of } 1.$

$K_f \subset G(\mathbb{A}_f)$, $S_{K_f}^G = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_{cos}^0 K_f$

"Cohomology of arithmetic grp" $H^i(S_{K_f}^G, M_{\lambda})$

Tools

Borel-Serre compactification. $\bar{S}_{K_f}^G = S_{K_f}^G \cup \partial S_{K_f}^G$, $\partial = \bigcup_p \partial_p$.

long exact sequence for the pair $(\bar{S}_{K_f}^G, \partial S_{K_f}^G)$

$\rightarrow H_c^i(S_{K_f}^G, M_{\lambda}) \xrightarrow{\tau} H^i(S_{K_f}^G, M_{\lambda}) \xrightarrow{\pi^i} H^i(\partial S_{K_f}^G, M_{\lambda}) \rightarrow \dots$

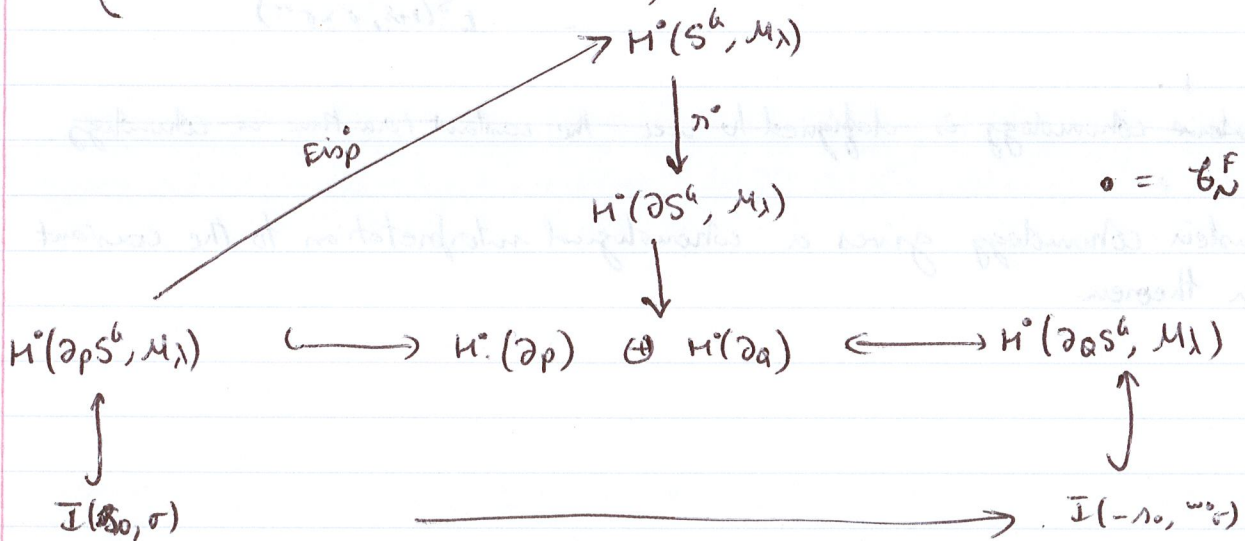
$H_{Eis}^i(S_{K_f}^G, M_{\lambda}) = \text{Im}(H^i(S_{K_f}^G, M_{\lambda}) \xrightarrow{\pi^i} H^i(\partial S_{K_f}^G, M_{\lambda}))$

\exists spectral seq. $\{E_1^{p,q}\} \Rightarrow H^i(\partial S_{K_f}^G, M_{\lambda})$

\hookrightarrow built from $H^i(\partial_p S_{K_f}^G, M_{\lambda})$

$H^i(\partial_p S_{K_f}^G, M_{\lambda}) = \bigoplus_{w \in W^p} \text{Ind}_p^G (H^{i-l(w)}(S_{K_f}^{M_p}, \tilde{M}_{w, \lambda}))$

$(W^p$ - Kostant repⁿ $W_G = W^p W_M$)



"Cohomological interpretation" takes this shape \uparrow

Theorems:

- ① Marnin-Drinfeld: $\left(\underbrace{I(\sigma_0, \sigma)}_{I(\sigma)} \oplus \underbrace{I(-\sigma_0, \sigma)}_{I(\sigma)} \right)$ splits off as a Hecke-summand from $H^0(\partial S^4, \mathcal{M}_1)$
- ② Image $\left(H^0(S^4, \mathcal{M}_1) \rightarrow H^0(\partial S^4, \mathcal{M}_1) \rightarrow I(\sigma) \oplus I(\omega_\sigma) \right)$ This image is like a line in a two dimensional space.
 $= \{ (\xi, T\xi) : \xi \in I(\sigma) \}$

The slope of this line is rational & after using transcendental means, I wig out. from
 Sheaf-theoretic $\text{slope} = \frac{1}{\Omega(\sigma)} \frac{L(-N/2, \sigma \times \sigma')}{L(-N/2, \sigma \times \sigma')} \text{ or } \frac{L(\sigma_0, \sigma)}{L(1+\sigma, \sigma)}$

§5 Ingredients

- compute the critical set of $L(s, \sigma \times \sigma')$ or $L(s, \sigma \times X)$ "Standard" "technique".
- Cohomology degrees & Weyl grp combinatorics. "Combinatorial Lemma".
- Manin-Drinfeld. (Hecke summands in Hecke cohomology, Jacquet-Shalika, Arthur.)
- Archimedean Problem.
 - Irreducibility of induced repⁿ $I_{P_0}^{\sigma_0}(\sigma_0, \sigma_0')$ (Speh, Speh-Vogan, Casselman-Shalika)
 - $T_{st, \alpha}$ is homothety by $\frac{L_{\sigma_0}(-N/2, \sigma_0 \times \sigma_0')}{L_{\sigma_0}(-N/2, \sigma_0 \times \sigma_0')}$
- Nonarchimedean / ramified problem.
 - $v \in S_0 \cup S_0'$, $T_{st, v}$ is a rational operator.

§6 (Time permitting).

t_n^F - lowest degree in which h_n/F has cusp. est. t_n^F - top degree.

~~all~~

$$t_n^F + t_{n'}^F + \frac{1}{2} \dim(N_p) = t_N^F$$

$$t_n^F + t_{n'}^F + \frac{1}{2} \dim(N_p) = t_N^F - 1.$$

Poincaré duality for $H^*(\partial S^h)$ is used.

Given $\sigma \in \mathcal{O}h(h_n/F, \mu)$ Look for $w \in W^p$ st $d(w) = \frac{1}{2} \dim(N_p)$

$\sigma' \in \mathcal{O}h(h_{n'}/F, \mu')$

& $w^{-1} \cdot (\mu + \mu')$ is dominant.

$$I_p^h(\sigma, \sigma') \hookrightarrow H^{t_N^F}(\partial S^h, M_\lambda)$$

$$w \cdot \lambda = \mu + \mu'.$$

$$W^p \xrightarrow{\sim} W^Q.$$

$$w \mapsto w'$$

$$l(w) + l(w') = \dim(N_p)$$

$\mathcal{C}(\mu, \mu')$ Combinatorial Lemma

(i) $\exists w \in W^p$ st. $l(w) = \frac{1}{2} \dim(N_p)$ & $w^{-1} \cdot (\mu + \mu')$ is dom.

\Leftrightarrow (ii) $-N/2, 1-N/2$ are critical for $L(s, \sigma \times \sigma')$ ($\sigma = h_n, \sigma' = h_{n'}, N = n+n'$)

or (ii') $-n$ & $1-n$ are critical for $L(s, \sigma \times \chi)$ ($\sigma = SO(n, n), \chi = h_{n'}$)

Move σ to $\sigma \otimes 1 - 1^m$ & let m vary over \mathbb{Z} , as long as $\mathcal{C}(\mu - m, \mu')$ is satisfied.

This gives exactly every pair of successive critical values no more & no less!

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