- Thank you for all your help & teaching
- Wish you a very happy birthday & many more years of beautiful mathematics.

1. Thank you for your help

   KHEILI MAMNOON AZ KOMAK-E-SHOMA.

2. Wish you a happy birthday

   TAVALOD-E-SHOMA MOBARAK

3. Wish you many more years of beautiful mathematics

   ~~SAAL SAAL-HAYE~~

   SAAL-HAYE ZIAD-E RIAZI-E-ZIBA

   BARAYETAN AREZOO DARAM.

Eisenstein Cohomology & Special values of L-fns.

§1 classical results:

Theorem (Manin, Shimura)

$\varphi \in S_k(\Gamma_0(N), \omega)_{\text{Prim}}$, $\exists\, u^{\pm}(\varphi) \in \mathbb{C}^{\times}$ s.t. — Periods

$\forall$ Dirichlet char. χ, $1 \le m \le k-1$ — critical set for $L(s, \varphi, \chi)$

$L_f(m, \varphi, \chi) \approx (2\pi i)^m \cdot u^{\pm}(\varphi) \cdot g(\chi)$; $\pm(-1)^m = \chi(-1)$

Cor: $1 \le m \le k-2$, $\dfrac{L_f(m, \varphi, \chi)}{L_f(m+1, \varphi, \chi)} \approx \dfrac{1}{(2\pi i)} \cdot \dfrac{u^{\pm}(\varphi)}{u^{\mp}(\varphi)}$

Cor: $\Omega(\varphi) = \dfrac{1}{i} \dfrac{u^{+}(\varphi)}{u^{-}(\varphi)}$ $\boxed{\dfrac{L(m, \varphi, \chi)}{L(m+1, \varphi, \chi)} \approx \Omega(\varphi)^{\varepsilon_m \varepsilon_\chi}}$ — $GL_n \times GL_{n'}/F$, n-even, n'-odd

Similar Theorems for Hilbert mod. forms. $GL_2 \times GL_1/F$

Theorem (Shimura ; Math. Ann. '77)

$\varphi \in S_k(\Gamma_0(N), \omega)$, $\xi \in S_\ell(\Gamma_0(N), \psi)$. Assume $k > \ell$.

$\ell \le m < k$ critical set.

$L_f(m, \varphi \times \xi) \approx (2\pi i)^{2m+1-\ell}\, u^{+}(\varphi) u^{-}(\varphi) \cdot g(\psi)$

Cor $\dfrac{L_f(m, \varphi \times \xi)}{L_f(m+1, \varphi \times \xi)} \approx \dfrac{1}{(2\pi i)^2}$ $\boxed{L(m, \varphi \times \xi) \approx L(m+1, \varphi \times \xi)}$

Similar Thm. for Hilbert mod. forms

$GL_2 \times GL_2$ — $GL_n \times GL_{n'}$, n, n'- both even. Harder + R

— $SO(n,n) \times GL_1$ (Bhagwat + R)

§2 Scope of the theorems on special values of L-fns. (via Eisenstein coh.)

Theorem ① (Harder + R)

σ - cusp. aut. repⁿ. of $GL_n(\mathbb{A}_F)$, F - tot. real, $\sigma \in Coh(GL_n/F, \mu)$

σ' - " " " " $GL_{n'}$, $\sigma' \in Coh(GL_{n'}/F, \mu')$ $\quad N = n + n'$

Assume a certain condition $\mathcal{C}(\mu, \mu')$ on μ & μ'.

- n - even, n' - odd. $\quad \dfrac{L(-N/2, \sigma\times\sigma')}{L(1-N/2, \sigma\times\sigma')} \approx \Omega(\sigma)^{\varepsilon_{\sigma'}}$.
- Can Replace σ by $\sigma \otimes |\cdot|^r$, $r \in \mathbb{Z}$ as long as $\mathcal{C}(\mu - r, \mu')$ is satisfied.

  If m & $m+1$ are critical then $\boxed{\dfrac{L(m, \sigma\times\sigma')}{L(1+m, \sigma\times\sigma')} \approx \Omega(\sigma)^{\varepsilon_{\sigma'}\varepsilon_m}}$

- n, n' both even, $\boxed{L(m, \sigma\times\sigma') \approx L(m+1, \sigma\times\sigma')}$.

"Theorem" ② (R)

σ, σ' as above, F - CM field $\quad \begin{matrix} F \\ | \\ F^+ \\ | \\ \mathbb{Q} \end{matrix} \quad \boxed{L(m, \sigma\times\sigma') \approx L(m+1, \sigma\times\sigma')}$

$\mathcal{C}(\mu, \mu')$ holds.

$m, m+1$ critical for $L(s, \sigma\times\sigma')$

Theorem ③ (Bhagwat + R)

σ - cusp. aut. repⁿ. of $SO(n,n)/_{F/\mathbb{Q}}$, F - tot. real, $\sigma \in Coh(SO(n,n)/_F, \mu)$

Assume, n - even.

- Ψ_σ - cusp.
- σ - globally generic.
- σ_∞ - d.s. repⁿ. H.C. para $\mu + \rho$. $\quad \mu = \mu_1 \geq \dots \geq \mu_{n-1} \geq |\mu_n|$ $\quad$ "$k > l$" Assume $|\mu_n| \geq 1$

χ - fin. order char of $\mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$

If $m, m+1 \in \{1 - |\mu_n|, \dots, |\mu_n| - 1, |\mu_n|\}$ then

$L(m, \sigma\times\chi) \approx L(m+1, \sigma\times\chi)$.

- My student Chaitanya Ambi $\quad SO(n,n)/$CM-field

Ultimate goal: Study arithmetic of Langlands-Shahidi L-fns.

§3 Analytic Theory of certain L-fns.

G - conn. red. grp / $\mathbb{Q}$.

P - max. parabolic $P = M_P N_P$, α - simple root deleted, $\tilde{\alpha}$ - fund wt.

σ - cusp. aut. repⁿ of $M(\mathbb{A})$ (assume σ - generic / not needed for many steps)

$f \in I_P^G(s\tilde{\alpha}, \sigma) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma \otimes s\tilde{\alpha}) \hookrightarrow \mathcal{C}^\infty(P(\mathbb{Q})\backslash G(\mathbb{A}))$

$\mathrm{Eis}_P(s, f)(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g) \in \mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$

Q - associate parabolic.

$\mathcal{F}_Q : \mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \longrightarrow \mathcal{C}(Q(\mathbb{Q})\backslash G(\mathbb{A}))$ $\quad \mathcal{F}_Q(\varphi)(g) = \int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} \varphi(ng)\,dn$

Langlands's Constant term theorem

$$\begin{tikzcd} & \mathcal{C}(G(\mathbb{Q})\backslash G(\mathbb{A})) \arrow[dr, "\mathcal{F}_Q"] & \\ \mathcal{C}^\infty(P(\mathbb{Q})\backslash G(\mathbb{A})) \arrow[ur, "\mathrm{Eis}"] & & \mathcal{C}(Q(\mathbb{Q})\backslash G(\mathbb{A})) \\ I_P^G(s,\sigma) \arrow[u, hook] \arrow[rr, "T_{st}(s)"] & & I_Q^G(-s, {}^{w_0}\sigma) \arrow[u, hook] \end{tikzcd}$$

$T_{st} = \bigotimes_v T_{st,v}$, $\quad v \notin S$, $\quad T_{st,v}(f_v^0) = \left(\prod_{i=1}^{m} \frac{L_v(is, \sigma_v, \tilde{\pi}_i)}{L_v(is+1, \sigma_v, \tilde{\pi}_i)}\right) f_v^0$

$G = GL_N$, $P = P_{(n,n')}$: $\quad T_{st}(s)$ will see, up to $v \in S_\infty$, $v \in S_{ram}$, $\dfrac{L^S(s, \sigma \times \sigma'^{\vee})}{L^S(1+s, \sigma \times \sigma'^{\vee})}$

~~Eisenstein cohomology is designed to see the constant term thm. in cohomology.~~

Eisenstein cohomology gives a cohomological interpretation to the constant term theorem.

§4 Eisenstein Cohomology

$G/\mathbb{Q} \supset B/\mathbb{Q} \supset T/\mathbb{Q} \supset Z/\mathbb{Q} \supset S/\mathbb{Q}$

$\lambda \in X^*_+(T)$

$M_{\lambda,E}$, alg. irred. rep^n. of $G \times E$ with h.w. λ.

C_∞ - max. cpct in $G(\mathbb{R})$, $\quad K_\infty = C_\infty \cdot S(\mathbb{R})$ $\quad K_\infty^0$ = conn. comp. of 1.

$K_f \subset G(\mathbb{A}_f)$, $\quad S^G_{K_f} = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty^0 K_f$

"Cohomology of arithmetic grps" $\quad H^\bullet(S^G_{K_f}, \tilde{M}_\lambda)$

Tools · Borel-Serre compactification. $\quad \bar{S}^G_{K_f} = S^G_{K_f} \cup \partial S^G_{K_f}$, $\partial = \bigcup_P \partial_P$.

· long exact sequence for the pair $(\bar{S}^G_{K_f}, \partial S^G_{K_f})$

$$\mathcal{H}^G_{K_f} \quad \longrightarrow H^\bullet_c(S^G_{K_f}, M_\lambda) \xrightarrow{i^\bullet} H^\bullet(S^G_{K_f}, M_\lambda) \xrightarrow{r^\bullet} H^\bullet(\partial S^G_{K_f}, M_\lambda) \longrightarrow \cdots$$

$$H^\bullet_{Eis}(S^G_{K_f}, M_\lambda) = \mathrm{Im}\left(H^\bullet(S^G_{K_f}, M_\lambda) \xrightarrow{r^\bullet} H^\bullet(\partial S^G_{K_f}, M_\lambda)\right)$$

· $\exists$ spectral seq. $\{E_1^{p,q}\} \Longrightarrow H^\bullet(\partial S^G_{K_f}, M_\lambda)$

↳ Built from $H^\bullet(\partial_P S^G_{K_f}, M_\lambda)$

· $H^\bullet(\partial_P S^G_{K_f}, M_\lambda) = \bigoplus_{w \in W^P} \mathrm{Ind}_P^G\left(H^{\bullet - \ell(w)}(S^{M_P}_{K_f^{M_P}}, \tilde{M}_{w\cdot\lambda})\right)$

(W^P Kostant rep^n. $W_G = W^P W_M$)

$$\begin{tikzcd}
 & & H^\bullet(S^G, M_\lambda) \arrow[d, "r^\bullet"] & & \\
 & & H^\bullet(\partial S^G, M_\lambda) \arrow[d] & & \\
H^\bullet(\partial_P S^G, M_\lambda) \arrow[uurr, "Eis_P^\bullet"] \arrow[rr, hook] & & H^\bullet(\partial_P) \oplus H^\bullet(\partial_Q) & & H^\bullet(\partial_Q S^G, M_\lambda) \arrow[ll, hook'] \\
I(\Lambda_0, \sigma) \arrow[u] \arrow[rrrr] & & & & I(-\Lambda_0, {}^{w_0}\sigma) \arrow[u]
\end{tikzcd}$$

$\bullet = b_N^F$

"cohomological interpretation" takes this shape

Theorems:

1. Manin-Drinfeld: $\Big(\underbrace{I(s_0,\sigma)}_{I(\sigma)} \oplus \underbrace{I(-s_0,{}^{w_0}\sigma)}_{I({}^{w_0}\sigma)}\Big)$ splits off as a Hecke-summand from $H^{\bullet}(\partial S^G, \mathcal{M}_\lambda)$

2. Image $\left(H^{\bullet}(S^G,\mathcal{M}_\lambda) \longrightarrow H^{\bullet}(\partial S^G,\mathcal{M}_\lambda) \longrightarrow I(\sigma)\oplus I({}^{w_0}\sigma) \right)$
$= \{ (\xi, T\xi) : \xi \in I(\sigma) \}$.

This image is like a line in a two dimensional space

The slope of this line is rational (sheaf-theoretic) & after using transcendental means & using aut. form

$$\text{slope} = \frac{1}{\Omega(\sigma)} \frac{L(-N/2, \sigma\times\sigma')}{L(1-N/2, \sigma\times\sigma')} \quad \text{or} \quad \frac{L(s_0,\sigma)}{L(1+s_0,\sigma)}$$

§5 Ingredients

- compute the critical set of $L(s, \sigma\times\sigma')$ or $L(s, \sigma\times\chi)$ "standard" "tedium".

- Cohomology degrees & Weyl grp combinatorics. "Combinatorial Lemma".

- Manin-Drinfeld (Hecke summands in bdry cohomology. Jacquet-Shalika, Arthur.)

- Archimedean Problem.
  - Irreducibility of induced repⁿ $I_{P_\infty}^{G_\infty}(s_0, \sigma_\infty)$ (Speh, Speh-Vogan, Casselman-Shahidi)
  - $T^{\bullet}_{st,\infty}$ is homothety by $\dfrac{L_\infty(-N/2, \sigma_\infty\times\sigma'_\infty)}{L_\infty(1-N/2, \sigma_\infty\times\sigma'_\infty)}$.

- Nonarchimedean / Ramified problem.
  $v\in S_\sigma \cup S_{\sigma'}$, $T_{st,v}$ is a rational operator.

§6 (Time permitting).

b_n^F - lowest degree in which GL_n/F has cusp. coh. t_n^F - top degree.

$$b_n^F + b_{n'}^F + \frac{1}{2}\dim(N_P) = b_N^F$$

$$t_n^F + t_{n'}^F + \frac{1}{2}\dim(N_P) = t_N^F - 1.$$

Poincaré duality for $H^\bullet(\partial S^h)$ is used

Given $\sigma \in \mathrm{Coh}(GL_n/F, \mu)$, $\sigma' \in \mathrm{Coh}(GL_{n'}/F, \mu')$. Look for $w \in W^P$ s.t. $l(w) = \frac{1}{2}\dim(N_P)$ & $w^{-1}\cdot(\mu+\mu')$ is dominant.

$$I_P^G(s_0, \sigma\times\sigma') \hookrightarrow H^{b_N^F}(\partial_P S^h, \mathcal{M}_\lambda) \qquad w\cdot\lambda = \mu * \mu'.$$

$$W^P \xrightarrow{\sim} W^Q. \qquad l(w) + l(w') = \dim(N^P)$$

$$w \longmapsto w'$$

$\mathcal{C}(\mu,\mu')$ Combinatorial Lemma

$$\boxed{\begin{array}{l} \text{(i)} \ \exists\, w \in W^P \text{ s.t. } l(w) = \frac{1}{2}\dim(N_P) \ \& \ w^{-1}\cdot(\mu+\mu') \text{ is dom.} \\ \Leftrightarrow \text{(ii)} \ -N/2,\ 1-N/2 \text{ are critical for } L(s, \sigma\times\sigma') \ (\sigma - GL_n,\ \sigma' - GL_{n'},\ N = n+n') \\ \text{or (ii)}' \ -n \ \& \ 1-n \text{ are crit. for } L(s, \sigma\times\chi) \ (\sigma - SO(n,n),\ \chi - GL_1) \end{array}}$$

Move σ to $\sigma \otimes |\cdot|^m$ & let m vary over $\mathbb{Z}$, as long as $\mathcal{C}(\mu - m, \mu')$ is satisfied. This gives exactly every pair of successive critical values no more & no less!

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