Functoriality and integral representations for $GSpin$

J. Cogdell

In a pair of papers in 2006 and 2014, Asgari and Shahidi showed the existence of a functorial lift from globally generic cuspidal representations π of $GSpin$ groups to GL_N. In the second paper, utilizing the descent from GL_N to $GSpin$ of Hundley and Sayag, they characterized the image representations Π on GL_N. Along the way, they needed a relation between the poles of the twisted L-function $L(s, \pi \times \tau)$, with τ a cuspidal representation of some GL_m, and an Eisenstein series on a $GSpin$ group induced from τ. This is precisely the type of relation one expects from a Rankin-Selberg type integral representation for $L(s, \pi \times \tau)$. Following the lead of Ginzburg, Rallis, and Soudry, we would like to explain our thoughts on these integral representations. This talk will be part survey of the past results of Asgari and Shahidi (functoriality) and part an explanation of work in progress with Asgari and Shahidi (integral representations).

Functoriality & Integral Representations for GSpin

Let k : no. field
$\psi : k\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$ non-trivial character

Let

$$G = G_n = \underbrace{GSpin_{2n+1},\ GSpin_{2n}}_{\text{split}},\ \text{or}\ \underbrace{GSpin^*_{2n}}_{\text{quasisplit}}.$$

(We will use $n' = 2n+1$ or $2n$ when needed).

Then

$$^L G = \begin{cases} GSp_{2n}(\mathbb{C}) \rtimes W_k & n' = 2n+1 \\ GSO_{2n}(\mathbb{C}) \rtimes W_k & n' = 2n \end{cases}$$

~~These are cases where G_D^0 is not classical but $^L G_D^0$ is.~~

~~Functoriality for~~

Associated to the map $\iota : {}^L G \hookrightarrow {}^L(GL_{2n})$ there should be a functorial lift or transfer from G to GL_{2n}. These are cases where G_D^0 is not classical but $^L G_D^0$ is; these are not currently approachable by endoscopy & trace formula. This functoriality was investigated by Asgari & Shahidi in 2 papers in 2006 & 2014.

Theorem (Existence).

Let $\pi = \otimes' \pi_v$ an irred. globally generic repn of $G(\mathbb{A})$
S : finite set of non-arch places of k where π_v and ψ_v are unramified.
For $v \notin S$, ϕ_v the Langlands parameter for π_v
$(\phi_v : W_{k_v} \longrightarrow {}^L G_v)$

Then there exists an automorphic repn
$$\Pi = \otimes' \Pi_v \quad \text{of } GL_{2n}(\mathbb{A})$$
s.th. for all $v \notin S$, Π_v is parametrized by
$$\Phi_v = \iota \circ \phi_v : W_{k_v} \longrightarrow GL_{2n}(\mathbb{C})$$
Furthermore $\omega_\Pi = \omega_\pi^n \mu$ w/ μ quadratic

and
$$\Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v} \quad \text{for } v \notin S. \qquad \boxtimes$$

The method of proof is now standard

- Use LLC for G and GL_{2n} to construct a local lift $\pi_v \longmapsto \Pi_v$ for $v \notin S$
- For $v \in S$ use the stability of γ to finesse the lack of LLC for G_v:
$$\gamma(s, \pi_v \times \eta_v, \psi_v) = \gamma(s, \Pi_v \times \eta_v, \psi_v)$$
for all η_v suff. highly ramified, depending only on central characters of both sides
- Use the converse theorem for GL_{2n} and the global control of $L(s, \pi\times\tau)$, $\varepsilon(s,\pi\times\tau)$ for τ cusp. aut. of $GL_m(\mathbb{A})$, $1 \le m \le 2n-2$, via Langlands - Shahidi method to obtain the automorphy of Π.

The second result is to characterize the image representation Π of $GL_{2n}(\mathbb{A})$.

<u>Theorem</u> (Characterization of Image)
Let π : irred. globally generic cusp. aut. rep of $G(\mathbb{A})$.
Then π has a unique functorial lift Π to an aut. repn of $GL_{2n}(\mathbb{A})$ s.t. $\Pi \simeq \tilde{\Pi} \otimes \omega_\pi$.
Moreover:

- Π is an isobaric sum $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_t$
$$= \mathrm{Ind}(\Pi_1 \times \cdots \times \Pi_t)$$
$$n_1 + \cdots + n_t = 2n$$

where
 (i) Each Π_i is a unitary cuspidal aut. repn of $GL_{n_i}(\mathbb{A})$
 (ii) $\Pi_i \not\simeq \Pi_j$ for $i \neq j$
 (iii) either
$$\begin{cases} L(s, \Pi_i, \wedge^2 \otimes \omega_\pi^{-1}) & n' = 2n+1 \\ L(s, \Pi_i, \mathrm{Sym}^2 \otimes \omega_\pi^{-1}) & n' = 2n \end{cases}$$
 has a pole at $s = 1$.

<u>Conversely</u> any aut repn $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_t$ satisfying (i) – (iii) is a functorial transfer of some π on $G(\mathbb{A})$. $\boxtimes$

There are 2 directions here
- characterization of the image
- exhaustion (the conversely direction)

- The <u>exhaustion</u> part is always done by a descent arguement, going from $GL_{2n} \to G$. For GSpin this was provided by Hundley & Sayag (2009, 2016)

- The direct characterization was provided by Asgari & Shahidi. Along the way, they needed the following result.

Let $H_m = GSpin_{2m}$ or $GSpin_{2m+1}$, split. $(m' = 2m, 2m+1)$

Let $P_m = M_m N_m$ be the Siegel parabolic sgp, so
$M_m \simeq GL_m \times GL_1$

Let τ: irred. unitary cusp. aut repn of $GL_m(\mathbb{A})$

η: ~~character~~ aut. character of $GL_1(\mathbb{A})$.

Let $\rho = \rho_{\tau,\eta,s} = \mathrm{Ind}_{P_m(\mathbb{A})}^{H_m(\mathbb{A})} (\tau|\cdot|^{s-1/2} \otimes \eta)$ $\qquad \eta = \omega_\pi^{-1}$

For a section f_s of ρ, form the Eisenstein series

$$E(h, f_s) = \sum_{\gamma \in P_m(k) \backslash H_m(k)} f_s(\gamma h)$$

<u>Proposition</u> If $L(s, \pi \times \tau)$ has a pole at $s = s_0$ w/ $\mathrm{Re}(s_0) \geq 1$, then for some choice of section f_s the Eisenstein series $E(h, f_s)$ has a pole at $s = s_0$.

- Let's not worry about what role this plays in the characterization of the image.
- If we think about this statement for a minute this is precisely the type of stmt. that one could expect from a Rankin-Selberg type integral representation for

$$L(s, \pi \times \tau)$$

involving $\varphi \in V_\pi$ a cusp form on $G(\mathbb{A})$

$E(h, f_s)$ an Eisenstein series on $H(\mathbb{A})$ induced from τ.

• When G-R-S proved the descent method needed to characterize the image of the lift from classical groups $\to GL_n$ they related this descent to certain integral repns for $G \times GL_m$. (In fact, the heuristics for descent came out of these integral representations.)

• In the descent work of Hundley & Sayag one can find an indication of the integral representations that are needed here.

We begin w/ the case where $m > n \geq 2$.
We need the two groups G_n + H_m to be of the "opposite parity" $\left(\begin{array}{ll} m' = 2m+1 \leftrightarrow n' = 2n \\ m' = 2m \leftrightarrow n' = 2n+1 \end{array}\right)$

Let $l = \begin{cases} m-n-1 & m' = 2m,\ n' = 2n+1 \\ m-n & m' = 2m+1,\ n' = 2n \end{cases}$

& for now require $l \geq 1$.

In Hundley + Sayag we find the following result.

<u>Propn</u> H_m has a parabolic sgp $Q_\ell = L_\ell N_\ell$ with

$$L_\ell = (GL_1)^\ell \times H_{m-\ell}$$

and there is a character Ψ_ℓ of N_ℓ s.th.

$$\mathrm{Stab}_{L_\ell}(\Psi_\ell)^0 = G_n$$

- If we now follow the method of GRS, we consider the Gelfand Graev period of the Eisenstein series

$$E^{(N_\ell, \Psi_\ell)}(h, f_s) = \int_{N_\ell(k)\backslash N_\ell(\mathbb{A})} E(uh, f_s)\, \Psi_\ell^{-1}(u)\, du$$

This is naturally an automorphic form on the G_n embedded as the stabilizer of (N_ℓ, Ψ_ℓ).

~~Step~~ Set

$$\mathcal{L}(\varphi, f_s) = \int_{Z_G(\mathbb{A})\, G(k)\backslash G(\mathbb{A})} \varphi(g)\, E^{(N_\ell, \Psi_\ell)}(g, f_s)\, dg$$

- This converges away from the poles of the Eisenstein series.

- One can now unfold this following the steps in G-R-S.

- When Hundley & Sayag pv'd descent for GSpin, they also followed the model of GRS + noted the following principles:
  - The unipotent varieties of H_m and SO_m are the same ($GSpin_{m'}$ and $SO_{m'}$)
  - The Weyl groups of H_m and SO_m are the same.

Modulo checking a few details related to these principles we have

<u>"Theorem"</u> (WIP w/ Asgari & Shahidi)

- $\mathcal{L}(\varphi, f_s) \neq 0 \Rightarrow \pi$ is globally generic (for a suitable Whittaker character)

- $\mathcal{L}(\varphi, f_s)$ is Eulerian, ie,

$$\mathcal{L}(\varphi, f_s) = \int_{N_G(\mathbb{A})Z_G(\mathbb{A})\backslash G(\mathbb{A})} W_\varphi^{\psi}(g) \int_{N_\ell(\mathbb{A}) \cap \beta^{-1} P_m(\mathbb{A}) \beta \backslash N_\ell(\mathbb{A})} f_s^{(Z_m, \psi)}(\beta u g)\, \Psi_\ell(u)^{-1}\, du\, dg$$

where

Z_m = max. unipotent sgp of $GL_m \subset P_m \subset H_m$.

β is a product of a Weyl gp elt and a rational diagonal element in GL_m

- This should factor as a product of local integrals

Based on preliminary calculations by Asgari we expect the following.

Expectation

$$L(\varphi, f_s) \sim \frac{L(s, \pi \times \tau)}{L(2s, \tau, R \otimes \omega_\pi^{-1})}$$

where $R = \begin{cases} \Lambda^2 & n' = 2n+1 \\ \mathrm{Sym}^2 & n' = 2n \end{cases}$.

- This would relate the poles of the L-function in $\mathrm{Re}(s) \geq 1$ to the poles of the Eisenstein series in the integral representation, as needed to characterize the image.

Remark For $m \leq n$, we would place the Gelfand - Graev period on the cusp form

$$\mathcal{L}(\varphi, f_s) = \int_{H(k) Z_H(\mathbb{A}) \backslash H(\mathbb{A})} \varphi^{(N_\ell, \Psi_\ell)}(h)\, E(h, f_s)\, dh.$$

The analysis of the unfolding is slightly more complicated here, but we expect it to work out the same.

- This will give the result needed to characterize the image