Answer Keys
for Example Problems
in Study Guide for Final Exam

I. 13. 2. 27.

P (1, 0, 5)
Q (2, 3, 9)

$\longrightarrow$ M : the midpoint of PQ

$$= \left(\frac{1+2}{2}, \frac{0+3}{2}, \frac{5+9}{2} \right)$$

$$= \left(\frac{3}{2}, \frac{3}{2}, 7 \right)$$

Eq. of the sphere.

$$\left(x - \frac{3}{2} \right)^2 + \left(y - \frac{3}{2} \right)^2 + (z-7)^2 = r^2$$

What is r (r^2)?

Passing P (or Q)

$\longrightarrow$

$$\left(1-\frac{3}{2}\right)^2+\left(0-\frac{3}{2}\right)^2+(5-7)^2 = r^2$$

$$\left(\text{or} \quad \left(2-\frac{3}{2}\right)^2+\left(3-\frac{3}{2}\right)^2+(9-7)^2 = r^2 \right)$$

$\longrightarrow$

$$\frac{26}{4}=r^2$$

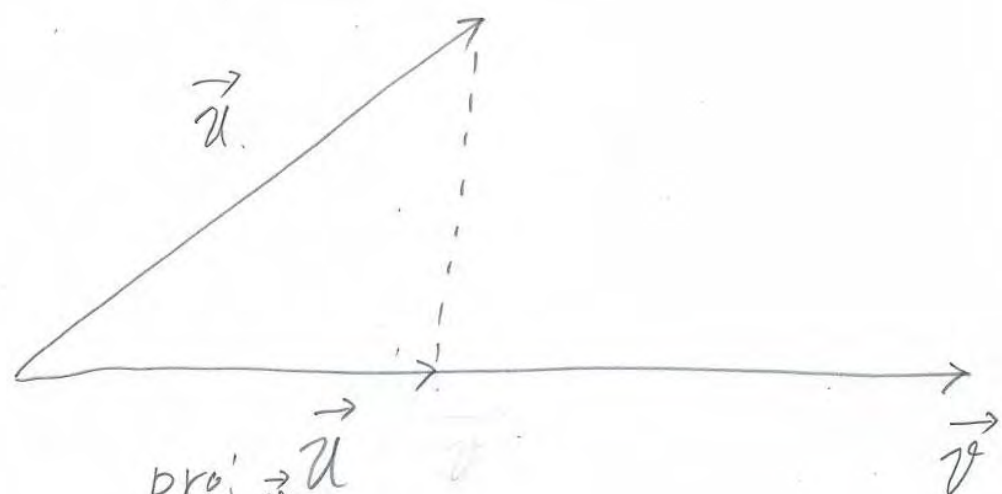
Final Answer :

Eq. of the sphere

$$\left(x-\frac{3}{2}\right)^2+\left(y-\frac{3}{2}\right)^2+(z-7)^2 = \frac{26}{4}$$

13.3 37.

$$\vec{u} = \langle -8, 0, 2 \rangle$$
$$\vec{v} = \langle 1, 3, -3 \rangle$$



$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right) v$$

$$= \frac{(-8)\cdot 1 + 0\cdot 3 + 2(-3)}{1^2 + 3^2 + (-3)^2} \langle 1, 3, -3 \rangle$$

$$= -\frac{14}{19} \langle 1, 3, -3 \rangle$$

$$\text{scal}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = -\frac{14}{\sqrt{19}}$$

13.4.30. ④

$\vec{u} = \langle -3, 0, 2 \rangle$

$\vec{v} = \langle 1, 1, 1 \rangle$

$$\vec{u} \times \vec{v} = \left\langle \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} -3 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} -3 & 0 \\ 1 & 1 \end{vmatrix} \right\rangle$$

$$= \langle -2, 5, -3 \rangle$$

the area of the parallelogram formed by $\vec{u}$ & $\vec{v}$

$$= |\vec{u} \times \vec{v}|$$

$$= \sqrt{(-2)^2 + 5^2 + (-3)^2}$$

$$= \sqrt{38}.$$

13. 4. 64

5

$\vec{u} = \langle 3, 1, 0 \rangle$

$\vec{v} = \langle 2, 4, 1 \rangle$

$\vec{w} = \langle 1, 1, 5 \rangle$

the volume of the parallelepiped formed by $\vec{u}, \vec{v}, \vec{w}$

determinant

$$= \left| \begin{vmatrix} 3 & 1 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 5 \end{vmatrix} \right|$$

absolute value

$$= \left| 3 \times \begin{vmatrix} 4 & 1 \\ 1 & 5 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} \right|$$

57

$$= \left| 3 \times 19 - 1 \times 9 + 0 \times (-2) \right|$$

$$= |48| = 48$$

13.5.20

(6)

$$\vec{u} = \langle 1, 1, -5 \rangle$$
$$\vec{v} = \langle 0, 4, 0 \rangle$$

$$\vec{u} \times \vec{v} = \left\langle \begin{vmatrix} 1 & -5 \\ 4 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & -5 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} \right\rangle$$

$$= \langle 20, 0, 4 \rangle$$

perpendicular both to $\vec{u}$ & $\vec{v}$

Choose $\vec{u} \times \vec{v}$ as the direction of the line.

Vector Eq.

$$\langle x, y, z \rangle = \langle -3, 4, 2 \rangle + t \langle 20, 0, 4 \rangle$$

t : the parameter

Parametric Eq.

$$x = -3 + 20t, \quad y = 4 + 0 \cdot t, \quad z = 2 + 4t$$

t : the parameter

Note:

Eq. in terms of x, y, z.

$$\begin{cases} \dfrac{x-(-3)}{20} = \dfrac{z-2}{4} \\ \& \quad y = 4 \end{cases}$$

Warning:

Do NOT write

$$\frac{x-(-3)}{20} = \frac{y-4}{0} = \frac{z-2}{4}.$$

2. <u>12.2.42.</u>

8

$$r = -2\cos\theta$$

$\times\ r.$

$$r^2 = -2r\cos\theta$$

$$\left\{ x^2 + y^2 = r^2, \quad \begin{matrix} x = r\cos\theta \\ y = r\sin\theta. \end{matrix} \right.$$

$$x^2 + y^2 = -2x$$

$$x^2 + 2x + y^2 = 0$$

$$(x^2 + 2x \underline{+1}) + y^2 = 0 \underline{+1}$$

$$(x+1)^2 + y^2 = 1^2$$

$\therefore$ a circle
with center $(-1, 0)$
& radius 1

12.2.43

9

$$r = 6\cos\theta + 8\sin\theta$$

$\times\ r$

$$r^2 = 6r\cos\theta + 8r\sin\theta$$

$$\left\{ x^2 + y^2 = r^2 \quad , \quad \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right.$$

$$x^2 + y^2 = 6x + 8y$$

$$x^2 - 6x + y^2 - 8y = 0$$

$$(x^2 - 6x + 9) + (y^2 - 8y + 16) = 0 + 9 + 16$$

$$(x-3)^2 + (y-4)^2 = 5^2$$

: a circle
with center $(3, 4)$
& radius 5.

12.2.47

$$r = 8 \sin\theta \qquad \times r$$

$$r^2 = 8 r \sin\theta$$

$$\left\{ x^2 + y^2 = r^2 , \quad \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right.$$

$$x^2 + y^2 = 8y$$

$$x^2 + y^2 - 8y = 0$$

$$x^2 + (y^2 - 8y + 16) = 0 + 16$$

$$x^2 + (y-4)^2 = 4^2$$

: a circle

with center $(0, 4)$

& radius 4.

12.2.48

11

$$r = \frac{1}{2\cos\theta + 3\sin\theta}$$

$$r(2\cos\theta + 3\sin\theta) = 1$$

$$2r\cos\theta + 3r\sin\theta = 1$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$2x + 3y = 1$$

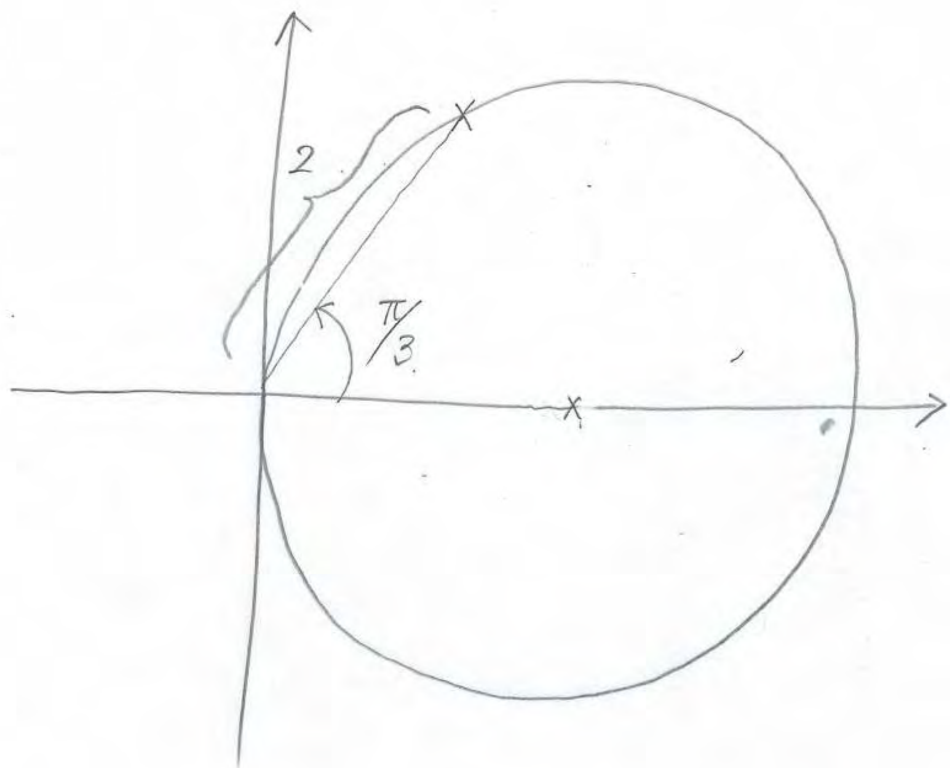
$$\longrightarrow \quad y = -\frac{2}{3}x + \frac{1}{3}$$

a line with slope $-\frac{2}{3}$ & y-intercept $\frac{1}{3}$

12.3.12.

12

$$r = 4\cos\theta \quad ; \quad \left(2, \frac{\pi}{3}\right)$$



$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

$$= \frac{(-4\sin\theta)\sin\theta + 4\cos\theta\cdot\cos\theta}{(-4\sin\theta)\cos\theta - (4\cos\theta)\sin\theta}$$

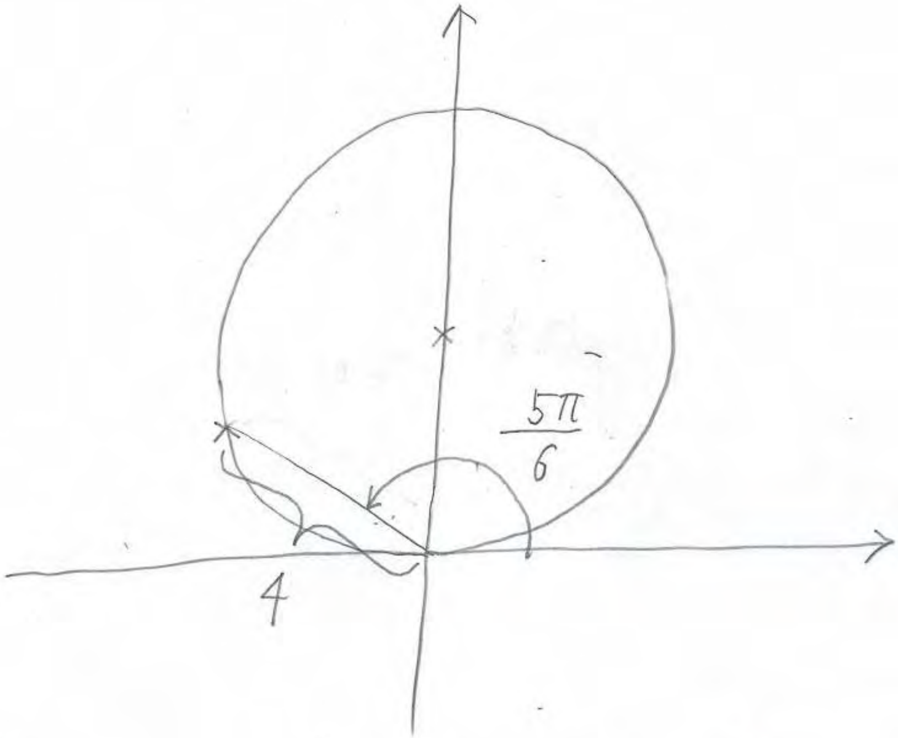
$$= \frac{-4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{-4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{-3+1}{-\sqrt{3}-\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

12 . 3 . 13

13

$$r = 8\sin\theta \quad ; \quad \left(4, \frac{5\pi}{6}\right)$$



$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

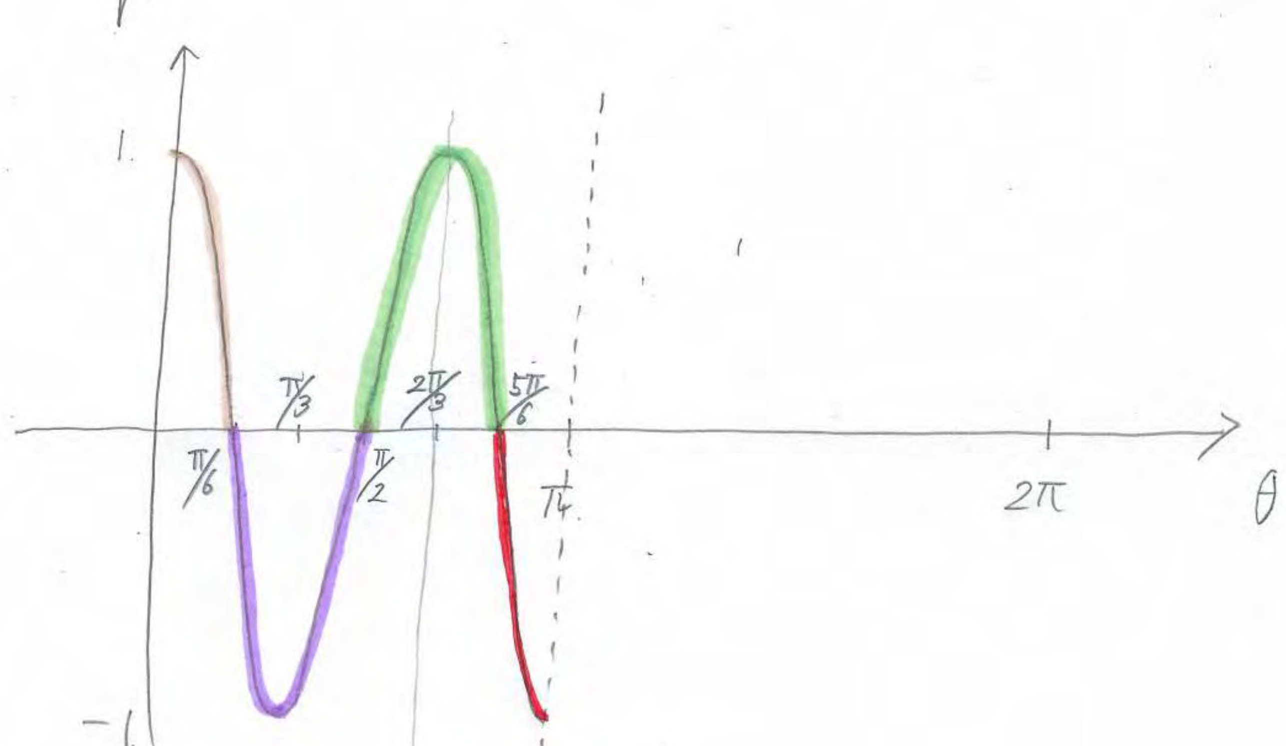
$$= \frac{(8\cos\theta)\sin\theta + (8\sin\theta)\cos\theta}{(8\cos\theta)\cos\theta - (8\sin\theta)\sin\theta}$$

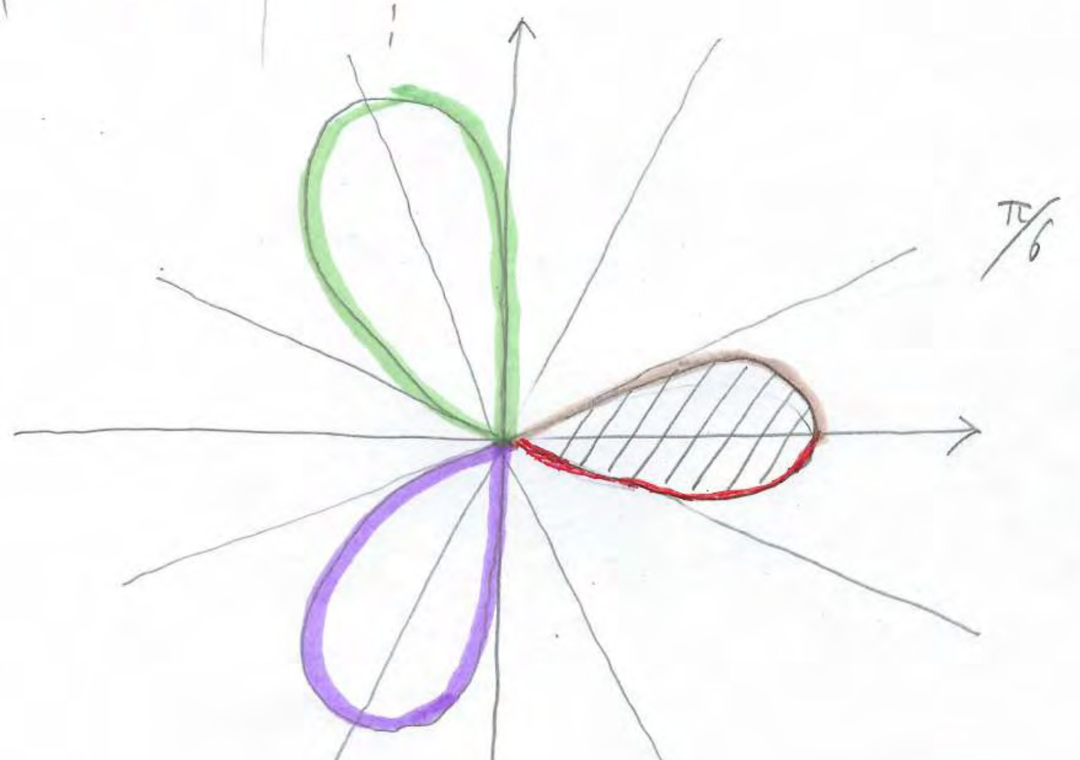
$$= \frac{8\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 8\left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)}{8\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) - 8\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$

$$= \frac{-\sqrt{3} - \sqrt{3}}{3 - 1} = -\sqrt{3}$$

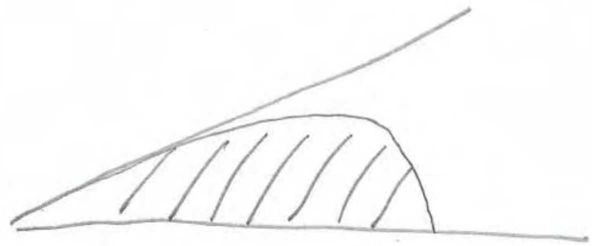
12 . 3 . 39

area of the region inside one leaf of $r = \cos 3\theta$





15



$$= \int_0^{\pi/6} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\pi/6} \frac{1}{2} \cos^2(3\theta) \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/6} \frac{1 + \cos(6\theta)}{2} \, d\theta$$

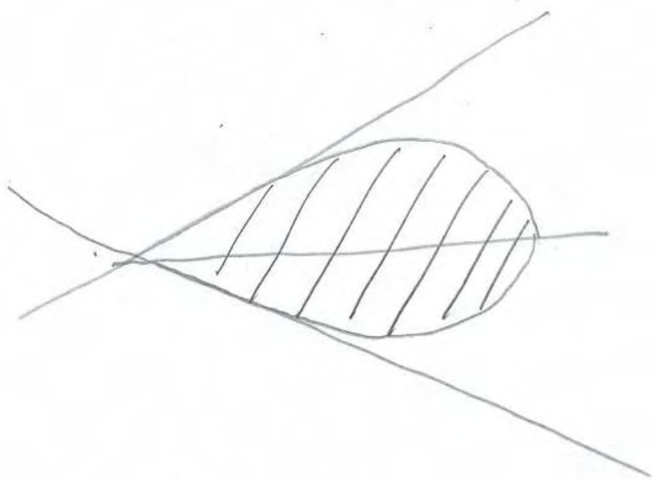
$$= \frac{1}{4} \left[\theta + \frac{1}{6} \sin(6\theta) \right]_0^{\pi/6}$$

$$= \frac{1}{4} \left[\left(\frac{\pi}{6} + \frac{1}{6} \cdot 0 \right) - \left(0 + \frac{1}{6} \cdot 0 \right) \right]$$

$$= \frac{\pi}{24}$$

$\longrightarrow$ area of the region inside of one leaf of $r = \cos 3\theta$.

$=$



$= \quad 2 \cdot \dfrac{\pi}{24}$

$= \dfrac{\pi}{12}.$

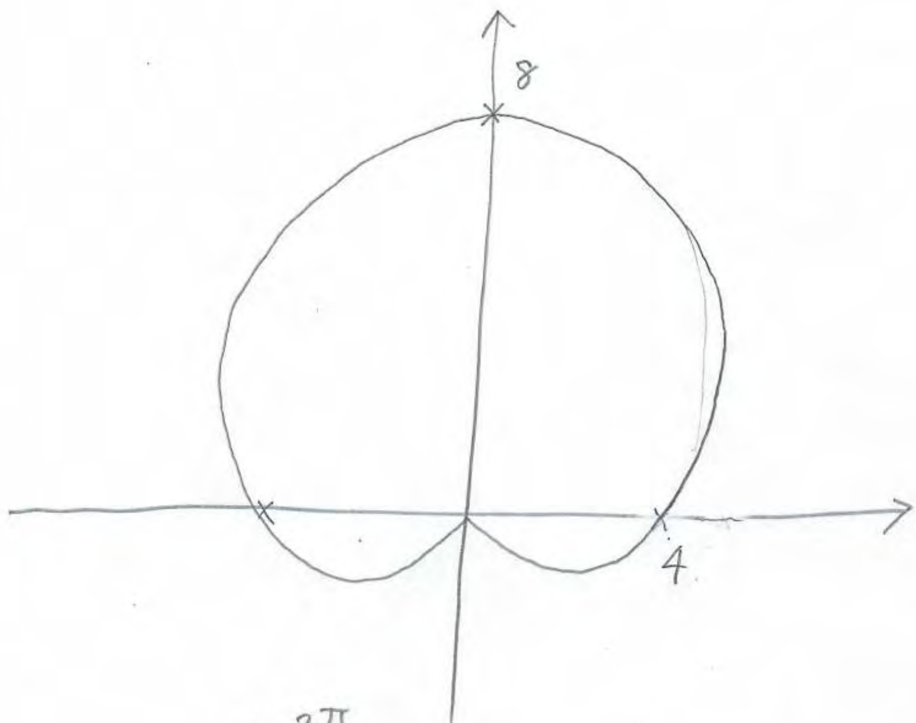
12.3.67.

17

the (arc) length of the complete cardioid

$$r = 4 + 4\sin\theta \quad \left(\to r' = \frac{dr}{d\theta} = 4\cos\theta \right)$$

$$0 \leqq \theta \leqq 2\pi$$



$$L = \int_0^{2\pi} \sqrt{(r')^2 + r^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(4\cos\theta)^2 + (4 + 4\sin\theta)^2}\, d\theta$$

$$= 4\int_0^{2\pi} \sqrt{\cos^2\theta + (1+\sin\theta)^2}\, d\theta$$ 18

$$= 4\int_0^{2\pi} \sqrt{\cos^2\theta + 1 + 2\sin\theta + \sin^2\theta}\, d\theta$$

$$= 4\int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

$$= 4\int_0^{2\pi} \sqrt{4\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}\, d\theta$$

Note: $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

$$2 + 2\sin\theta = 4\left\{\frac{1+\sin\theta}{2}\right\}$$

$$= 4\left\{\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)}{2}\right\}$$

$$= 4\cos^2\left(\frac{\frac{\pi}{2}-\theta}{2}\right)$$

$$= 4\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$= 4 \int_0^{2\pi} \left| 2\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \right| d\theta$$

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$$= 4 \left\{ \int_0^{\frac{3\pi}{2}} 2\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) d\theta \right.$$

$$\left. + \int_{\frac{3\pi}{2}}^{2\pi} -2\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) d\theta \right\}$$

$$= 4 \cdot \left\{ \left[2(-2)\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \right]_0^{3\pi/2} \right.$$

$$\left. - \left[2(-2)\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \right]_{3\pi/2}^{2\pi} \right\}$$

$$= 4 \left\{ 2(-2)\left[(-1) - \frac{\sqrt{2}}{2} \right] \right.$$

$$\left. - 2(-2)\left[\left(-\frac{\sqrt{2}}{2}\right) - (-1) \right] \right\}$$

$$= 32.$$

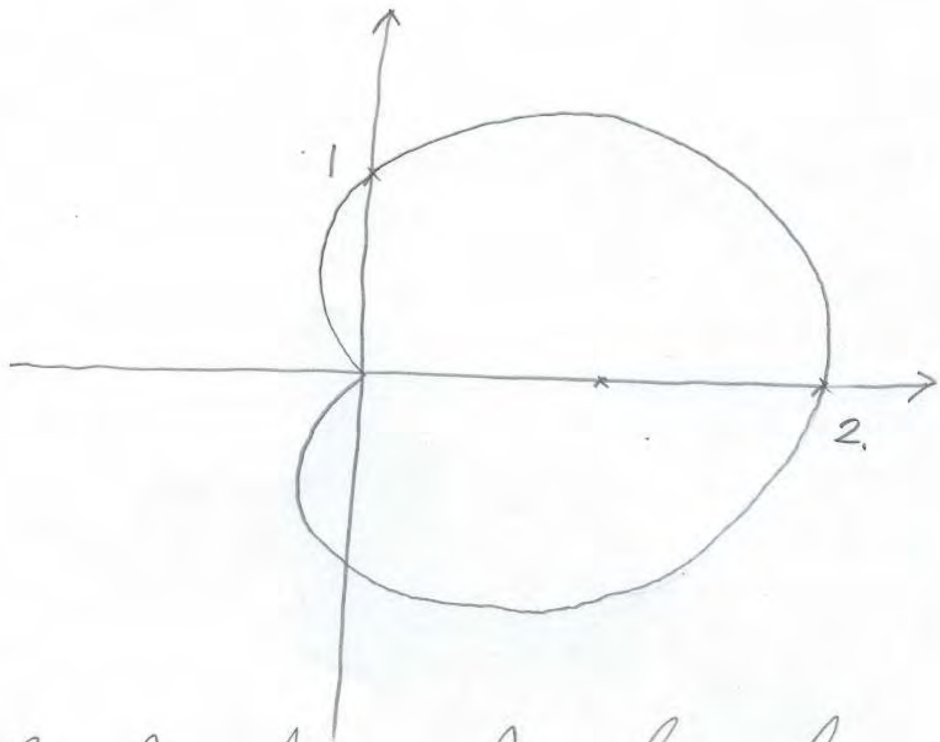
Remark.

It is much more straightforward to compute the (arc) length of the complete cardioid

$$r = 4 + 4\cos\theta$$

$$0 \leq \theta \leq 2\pi$$

$$(\rightarrow r' = -4\sin\theta)$$



This has the same length as the previous one.

$$L = \int_0^{2\pi} \sqrt{(r')^2 + r^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(-4\sin\theta)^2 + (4 + 4\cos\theta)^2}\, d\theta$$

$$= 4 \int_0^{2\pi} \sqrt{\sin^2\theta + (1+\cos\theta)^2}\, d\theta$$

$$= 4 \int_0^{2\pi} \sqrt{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}\, d\theta$$

$$= 4 \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta$$

$$\left(\text{Note: } 2 + 2\cos\theta = 4 \left\{ \frac{1+\cos\theta}{2} \right\} = 4 \cdot \cos^2\left(\frac{\theta}{2}\right) \right)$$

$$= 4 \int_0^{2\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$= 4 \int_0^{2\pi} \left| 2 \cos\left(\frac{\theta}{2}\right) \right| d\theta$$

$$= 4 \left\{ \int_0^{\pi} 2 \cos\left(\frac{\theta}{2}\right) d\theta + \int_{\pi}^{2\pi} -2 \cos\left(\frac{\theta}{2}\right) d\theta \right\}$$

$$= 4 \left\{ \left[4 \sin\left(\frac{\theta}{2}\right) \right]_0^{\pi} + \left[-4 \sin\left(\frac{\theta}{2}\right) \right]_{\pi}^{2\pi} \right\}$$

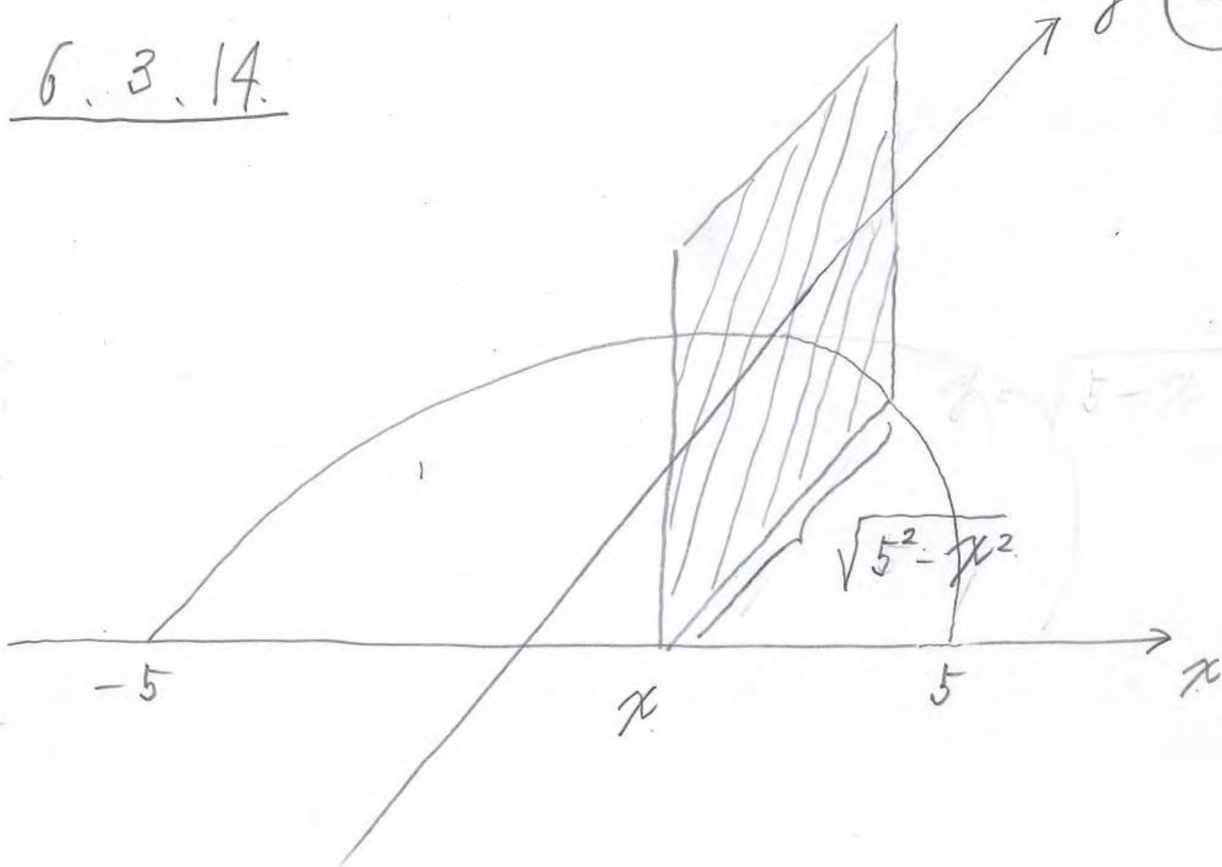
$$= 4 \left\{ [4 - 0] + [-4 \cdot 0 - (-4 \cdot 1)] \right\}$$

$$= 4 \cdot 8 = 32.$$

$\lrcorner$

3.

6.3.14.



$$V = \int_a^b A(x)\,dx$$

$$= \int_{-5}^{5} \left(\sqrt{5^2 - x^2}\right)^2 dx$$

$$= \int_{-5}^{5} (5^2 - x^2)\, dx$$

$$= \left[25x - \frac{x^3}{3} \right]_{-5}^{5} = \frac{500}{3}$$

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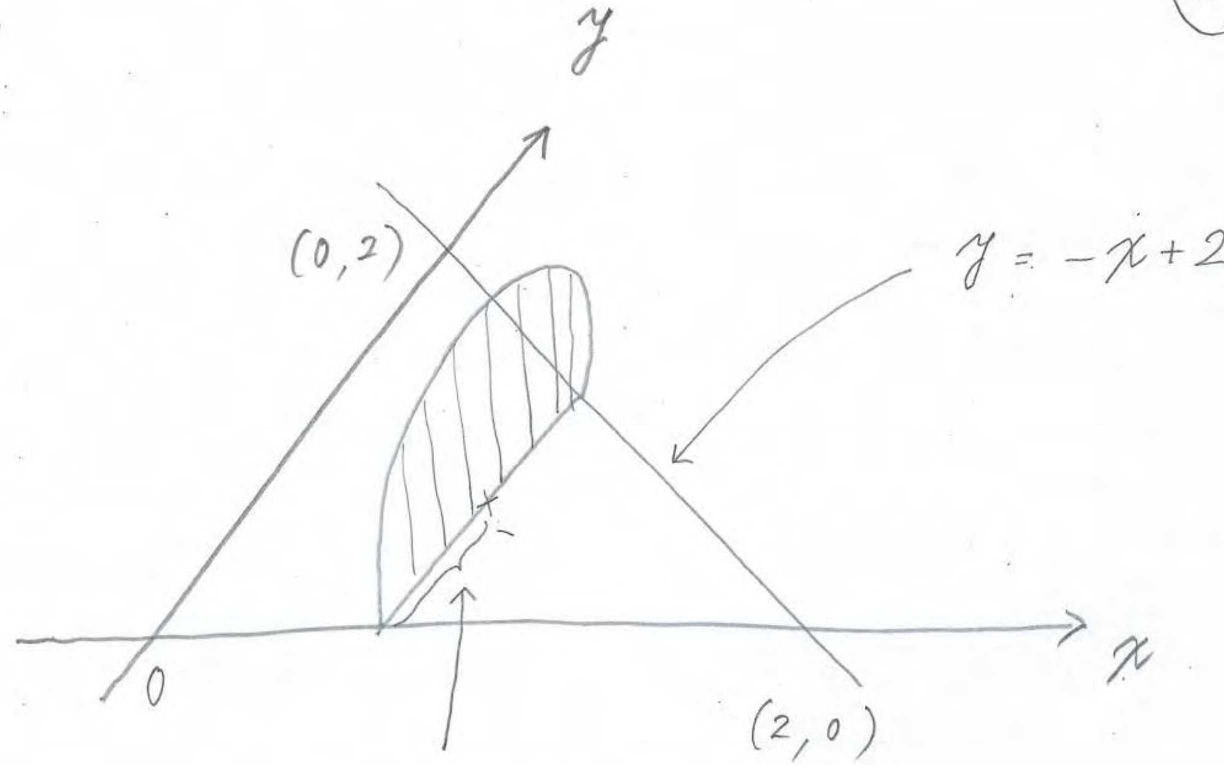
6.3.15.



$\frac{1}{2}(-x+2)$

$$V = \int_a^b A(x)\, dx$$

$$= \int_0^2 \frac{1}{2}\, \pi \left\{ \frac{1}{2}(-x+2) \right\}^2 dx$$

$$= \frac{\pi}{8} \int_0^2 (-x+2)^2\, dx$$

$$= \frac{\pi}{8} \int_0^2 (x^2 - 4x + 4)\, dx$$

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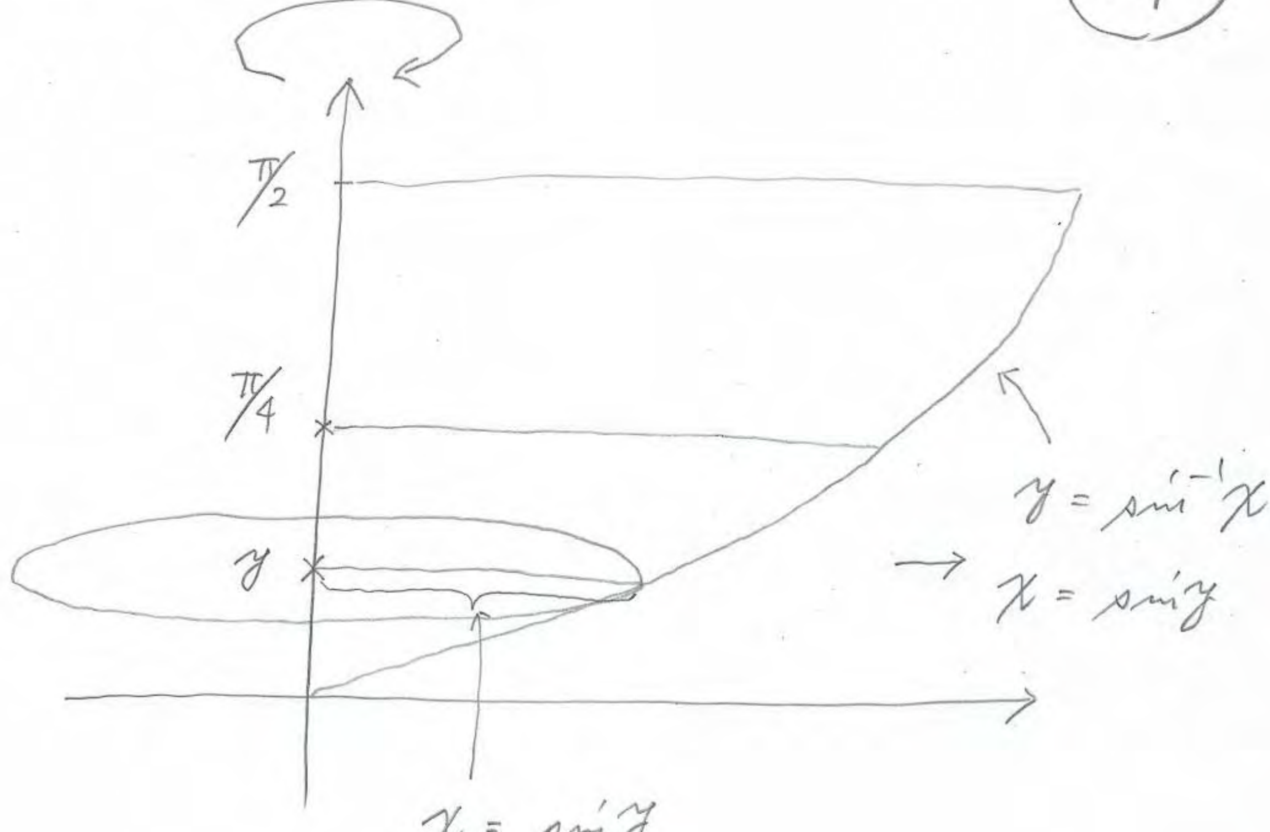
$$= \frac{\pi}{8} \left[\frac{x^3}{3} - 2x^2 + 4x \right]_0^2 = \frac{\pi}{3}$$

6.3.33



$$V = \int_0^{\pi/4} \pi (\sin y)^2 \, dy$$

$$= \pi \int_0^{\pi/4} \sin^2 y \, dy$$

$$= \pi \int_0^{\pi/4} \frac{1 - \cos 2y}{2} \, dy$$

$$= \frac{\pi}{2}\left[y - \frac{1}{2}\sin 2y \right]_0^{\pi/4}$$

$$= \frac{\pi}{2}\left[\left(\pi/4 - \frac{1}{2}\cdot 1\right) - \left(0 - \frac{1}{2}\cdot 0\right) \right]$$

$$= \frac{\pi}{8}(\pi - 2)$$

6 . 3 . 36

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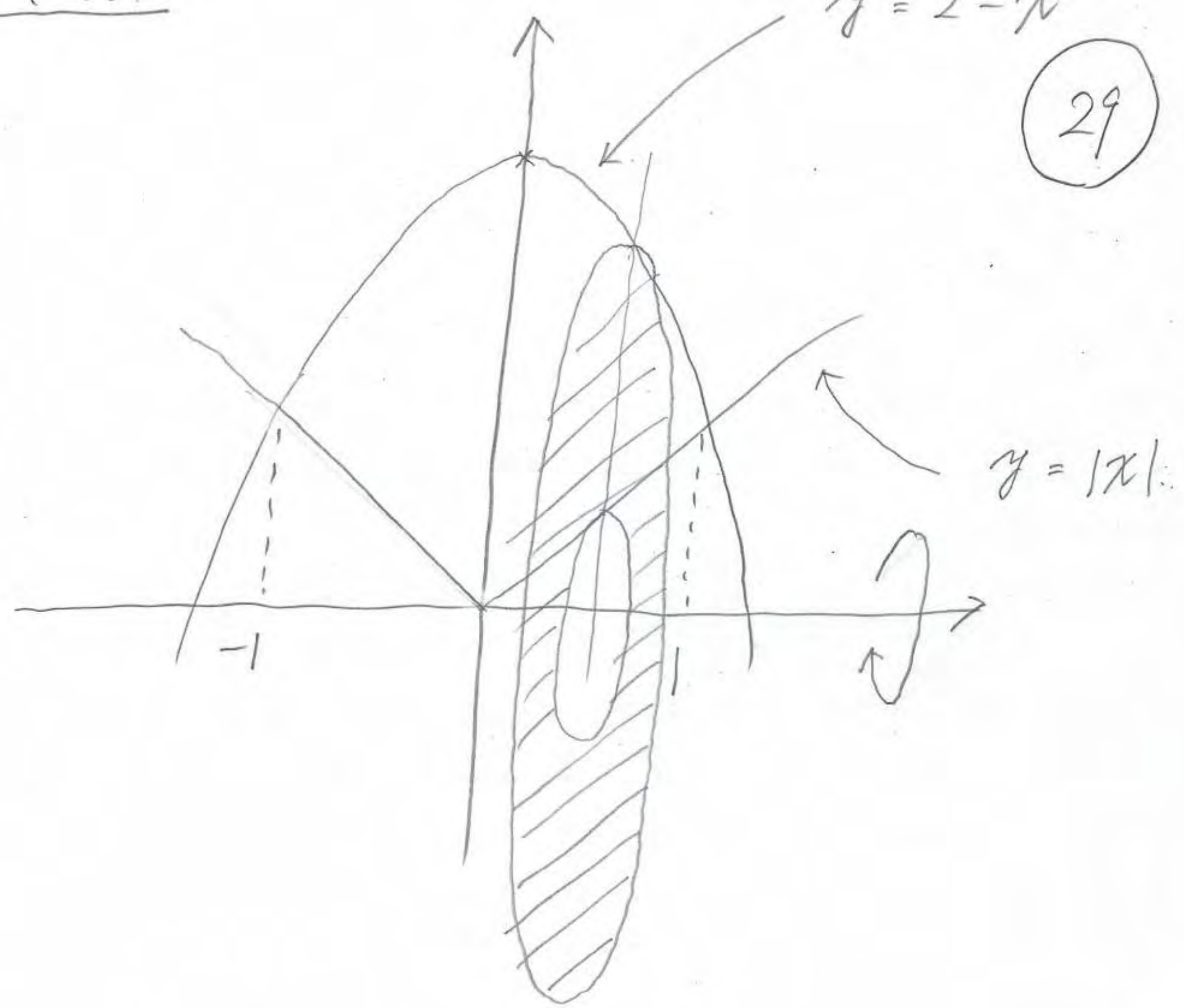

$$V = \int_{-1}^{1} \{ \pi (2-x^2)^2 - \pi |x|^2 \} \, dx$$

$$= \pi \int_{-1}^{1} \{ (2-x^2)^2 - |x|^2 \} \, dx$$

$$= \pi \int_{-1}^{1} (x^4 - 5x^2 + 4) \, dx$$

$$= \pi \left[\frac{x^5}{5} - 5 \cdot \frac{x^3}{3} + 4x \right]_{-1}^{1}$$

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$$= \frac{76}{15} \pi.$$

6.3.37

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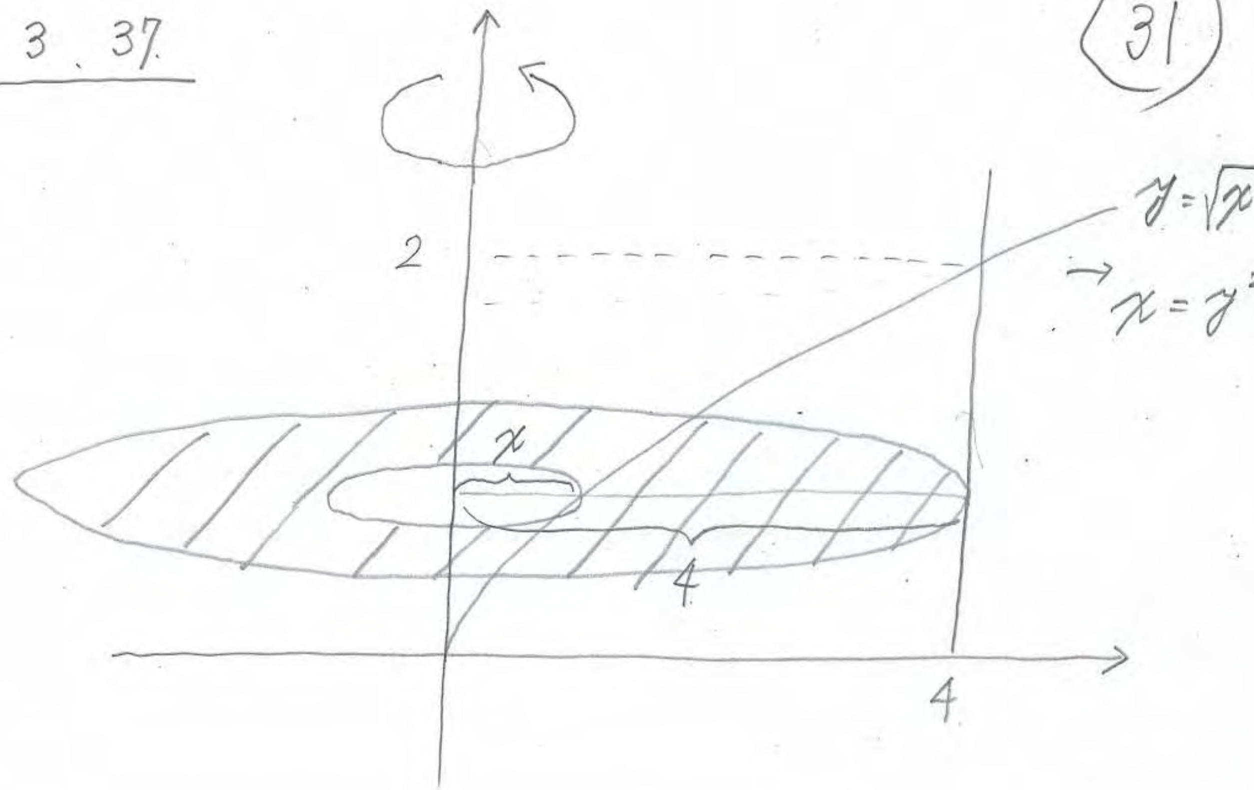


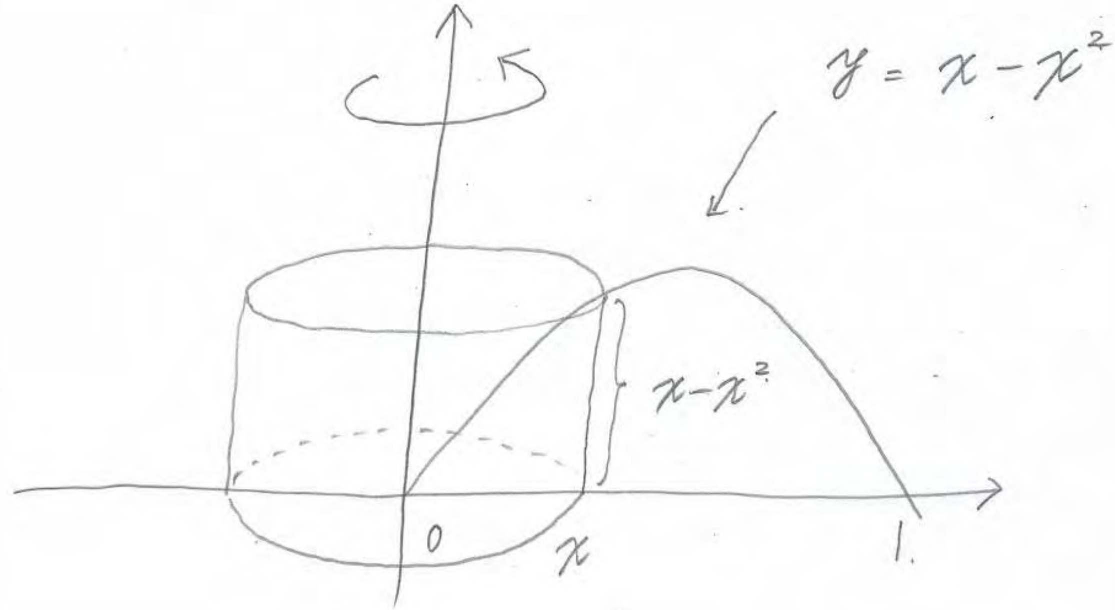
$$V = \int_0^2 \{\pi 4^2 - \pi (y^2)^2\}\, dy$$

$$= \pi \int_0^2 \{16 - y^4\}\, dy$$

$$= \pi \left[16y - \frac{y^5}{5} \right]_0^2$$

$$= \frac{128}{5}\pi.$$

6.4.9.



$\times \; 1 \; \times \; dx$

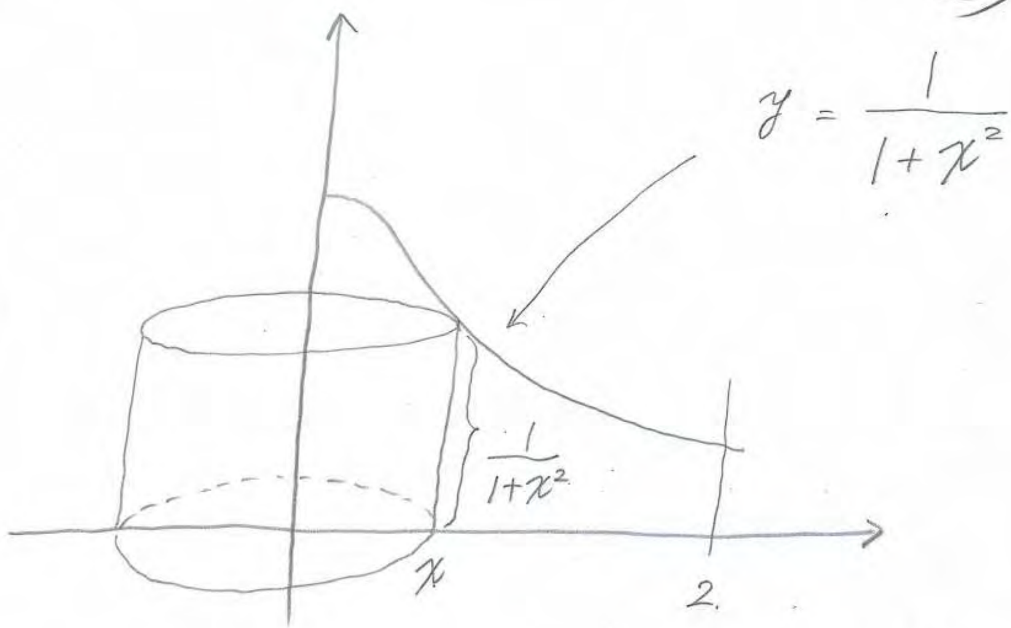
$$V = \int_0^1 2\pi x \cdot (x - x^2) \cdot dx$$

$$= 2\pi \int_0^1 x(x - x^2)\, dx$$

$$= 2\pi \int_0^1 (x^2 - x^3)\, dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\pi}{6}$$

6.4.10



$$\times \quad 1 \quad \times \quad dx$$

$$V = \int_0^2 2\pi x \cdot \frac{1}{1+x^2} \cdot dx$$

$$= 2\pi \int_0^2 \frac{x}{1+x^2}\, dx$$

$$\left(\begin{array}{ll} x & u = 1+x^2 \\ 2 & 5 \\ 0 & 1 \end{array} \quad du = 2x\,dx \right)$$

$$= 2\pi \int_1^5 \frac{1}{u} \left(\frac{1}{2}\, du\right)$$

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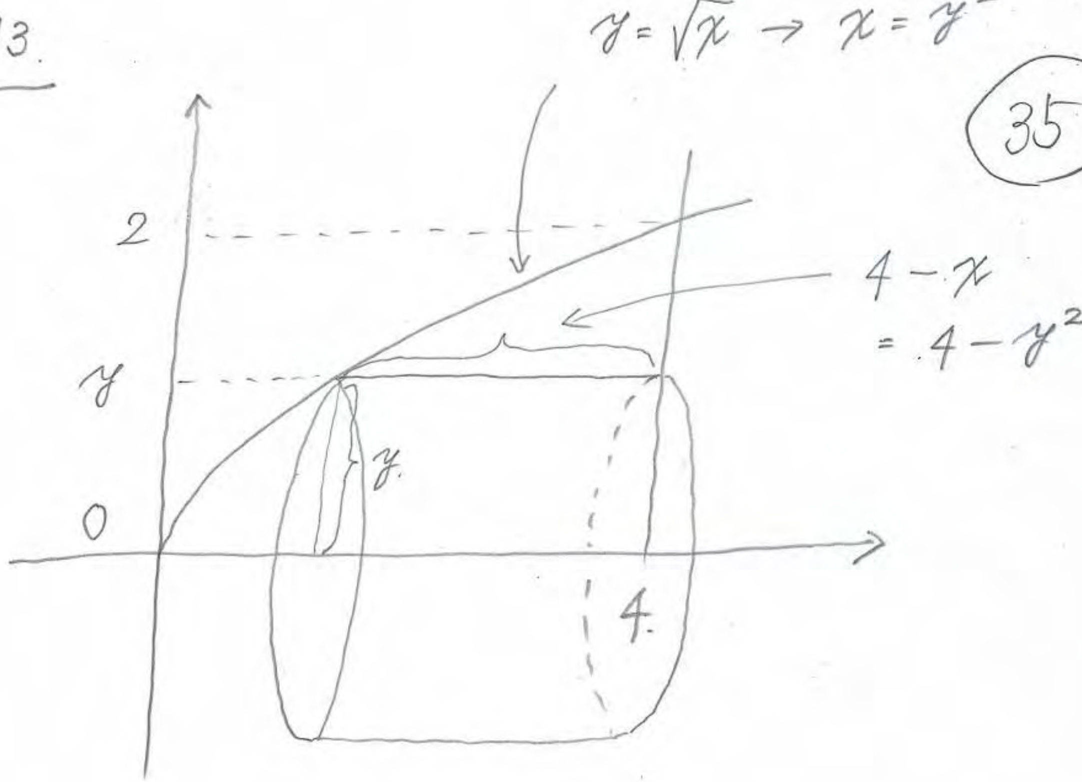
$$= \pi \left[\ln u \right]_1^5$$

$$= \pi \left[\ln 5 - \ln 1 \right]$$

$$= \pi \cdot \ln 5.$$

6.4.13.



$$V = \int_0^2 2\pi y\ (4-y^2)\, dy$$

$$= 2\pi \int_0^2 y\ (4-y^2)\, dy$$

$$= 2\pi \int_0^2 (4y - y^3)\, dy$$

$$= 2\pi \left[4 \cdot \frac{y^2}{2} - \frac{y^3}{3} \right]_0^2 = \frac{32}{3}\pi$$

4.

6.6.17

$$y = \frac{1}{4}\left(e^{2x}+e^{-2x}\right)$$

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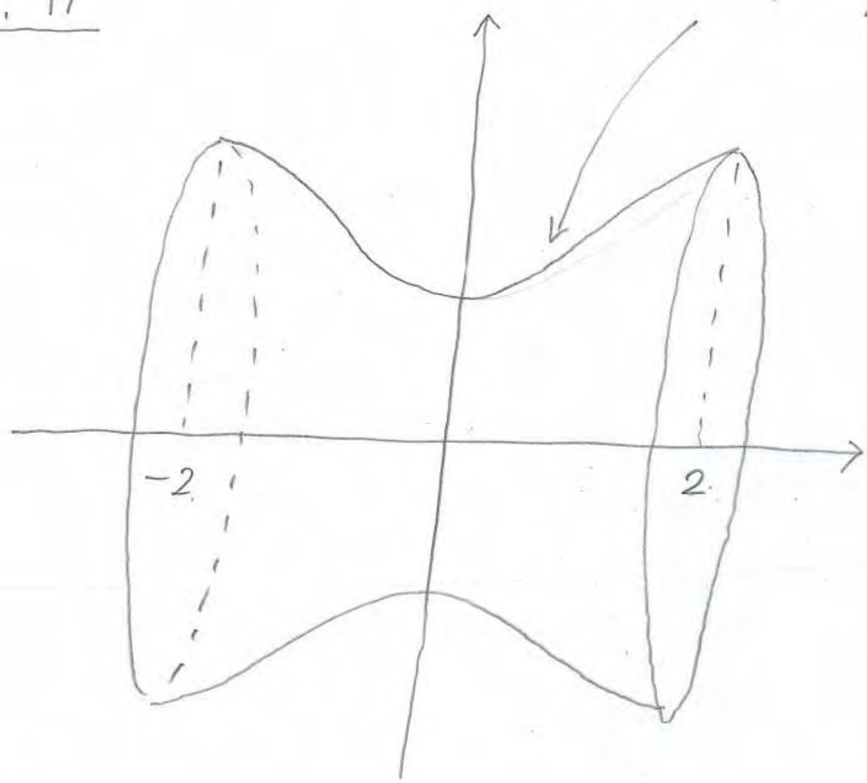

$$S = \int_{-2}^{2} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\left(\begin{aligned} \frac{dy}{dx} &= \frac{1}{4}\left(2e^{2x}-2e^{-2x}\right) \\ &= \frac{1}{2}\left(e^{2x}-e^{-2x}\right) \end{aligned} \right)$$

$$= \int_{-2}^{2} 2\pi \cdot \left\{\frac{1}{4}\left(e^{2x}+e^{-2x}\right)\right\} \sqrt{1+\left\{\frac{1}{2}\left(e^{2x}-e^{-2x}\right)\right\}^2}\, dx$$

Computation of what's inside of $\sqrt{\ }$:

(37)

$$1 + \left\{\frac{1}{2}\left(e^{2x} - e^{-2x}\right)\right\}^2$$

$$= 1 + \frac{1}{4}\left\{\left(e^{2x}\right)^2 - 2 + \left(e^{-2x}\right)^2\right\}$$

$$= \frac{1}{4}\left\{\left(e^{2x}\right)^2 + 2 + \left(e^{-2x}\right)^2\right\}$$

$$= \left\{\frac{1}{2}\left(e^{2x} + e^{-2x}\right)\right\}^2$$

$$= \int_{-2}^{2} 2\pi \left\{\frac{1}{4}\left(e^{2x} + e^{-2x}\right)\right\}\left\{\frac{1}{2}\left(e^{2x} + e^{-2x}\right)\right\} dx$$

$$= \frac{\pi}{4}\int_{-2}^{2}\left(e^{2x} + e^{-2x}\right)^2 dx$$

$$= \frac{\pi}{4}\int_{-2}^{2}\left(e^{4x} + 2 + e^{-4x}\right) dx$$

$$= \frac{\pi}{4}\left[\frac{1}{4}e^{4x} + 2x - \frac{1}{4}e^{-4x}\right]_{-2}^{2}$$

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$$= \frac{\pi}{4}\left[\left(\frac{1}{4}e^{8} + 4 - \frac{1}{4}e^{-8}\right) - \left(\frac{1}{4}e^{-8} - 4 - \frac{1}{4}e^{8}\right)\right]$$

$$= \frac{\pi}{4}\left[\frac{1}{2}e^{8} + 8 - \frac{1}{2}e^{-8}\right]$$

$$= \frac{\pi}{8}\left[e^{8} + 16 - e^{-8}\right]$$

□

12.6.33

39

$$y = \frac{x^3}{3} + \frac{1}{4x}$$

$$\frac{1}{2} \le x \le 2.$$

about x-axis

$$S = \int_{\frac{1}{2}}^{2} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\left(\frac{dy}{dx} = x^2 - \frac{1}{4x^2} \right)$$

$$= \int_{\frac{1}{2}}^{2} 2\pi \left(\frac{x^3}{3} + \frac{1}{4x}\right) \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\, dx$$

Computation of what's inside of $\sqrt{}$:

40

$$1 + \left(x^2 - \frac{1}{4x^2}\right)^2$$

$$= 1 + (x^2)^2 - 2x^2 \frac{1}{4x^2} + \left(\frac{1}{4x^2}\right)^2$$

$$= 1 + (x^2)^2 - \frac{1}{2} + \left(\frac{1}{4x^2}\right)^2$$

$$= (x^2)^2 + \frac{1}{2} + \left(\frac{1}{4x^2}\right)^2$$

$$= (x^2)^2 + 2 \cdot x^2 \cdot \frac{1}{4x^2} + \left(\frac{1}{4x^2}\right)^2$$

$$= \left(x^2 + \frac{1}{4x^2}\right)^2$$

$$= \int_{\frac{1}{2}}^{2} 2\cdot\pi \left(\frac{x^3}{3} + \frac{1}{4x}\right)\left(x^2 + \frac{1}{4x^2}\right) dx$$

$$= 2\pi \int_{\frac{1}{2}}^{2} \left(\frac{x^3}{3} + \frac{1}{4x}\right)\left(x^2 + \frac{1}{4x^2}\right) dx$$

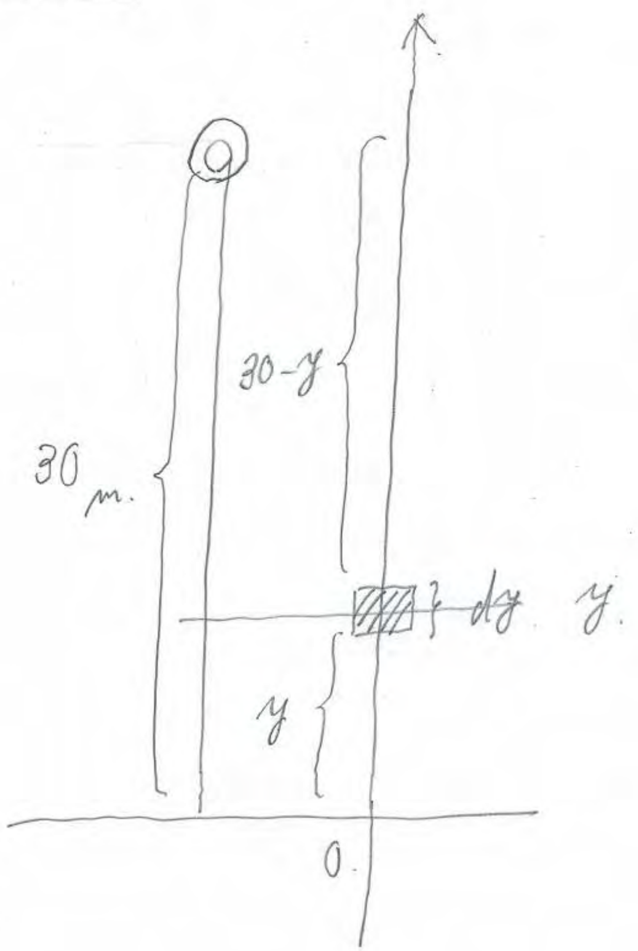
$$= 2\pi \int_{\frac{1}{2}}^{2} \left(\frac{x^5}{3} + \frac{x}{3} + \frac{1}{16x^3} \right) dx$$

$$= 2\pi \left[\frac{1}{3}\frac{x^6}{6} + \frac{1}{3}\frac{x^2}{2} + \frac{1}{16}\left(-\frac{1}{2x^2}\right) \right]_{\frac{1}{2}}^{2}$$

$$= 2\pi \left[\left\{ \frac{1}{3}\cdot\frac{64}{6} + \frac{1}{3}\cdot 2 - \frac{1}{16\cdot 8} \right\} - \left\{ \frac{1}{3}\,\frac{1}{6\cdot 64} + \frac{1}{3}\cdot\frac{1}{8} - \frac{1}{8} \right\} \right]$$

$$= \frac{9961}{1152}\pi.$$

6.7.31



5 kg/m density

We use $g = 9.8\ m/s^2$ for the acceleration due to gravity.

a. ∘ mass of ▨ part of the chain

$5\,dy$

∘ weight of ▨ part of the chain

$$g \times 5\,dy = 5g\,dy$$

- work to lift part of the chain to the cylinder

$$(30 - y) \times 5g\,dy$$

$$= 5g(30 - y)\,dy$$

- Total work

$$\int_0^{30} 5g(30 - y)\,dy$$

$$= 5g \int_0^{30} (30 - y)\,dy$$

$$= 5g \left[30y - \frac{y^2}{2} \right]_0^{30}$$

$$= 49 \times \frac{30^2}{2}$$

$$= 22050 \text{ J}$$

b Work to be done in Part b.

= Work done in Part a

\+ Work to lift the 50 kg block

$= 22050 + 30 \times 9.8 \times 50$

$= 22050 + 14700$

$= 36750$ J

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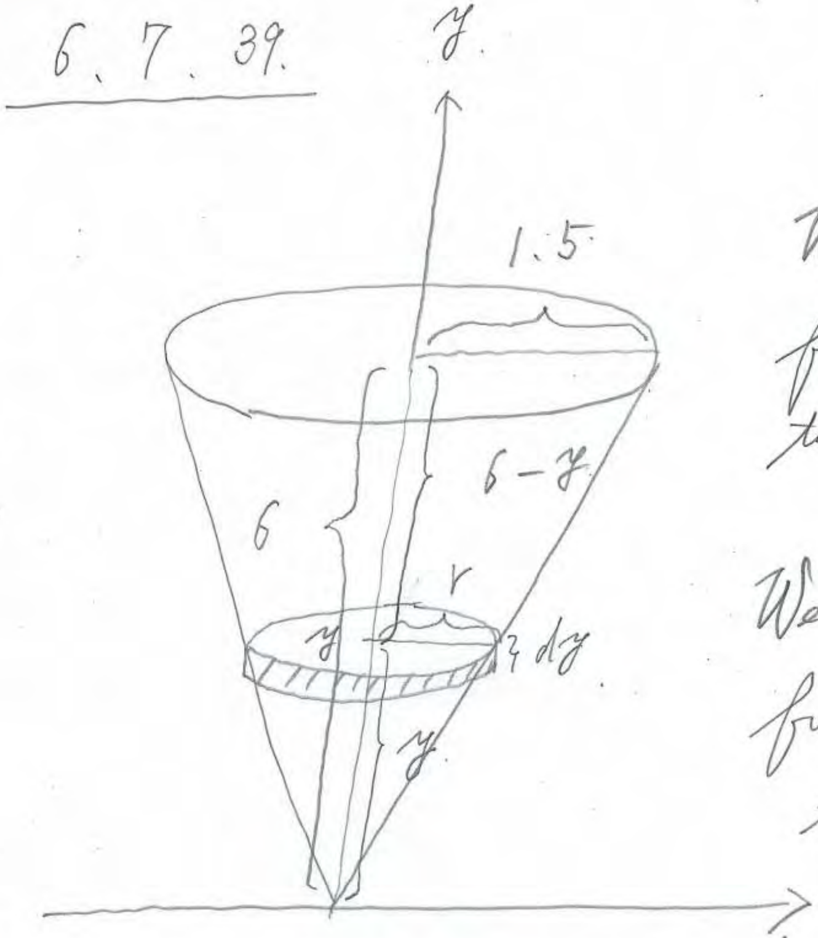
6.7.39.



We use $g = 9.8\ m/s^2$ for the acceleration due to gravity

We use $\rho = 1000\ kg/m^3$ for the density of the water.

a.

- volume of the slice  of the water at level y.

$$= \pi r^2 \cdot dy \qquad r = \frac{1.5}{6} y = \frac{1}{4} y$$

$$\pi \left(\frac{1}{4} y\right)^2 dy$$

$$= \frac{\pi}{16} y^2 dy$$

- mass of the slice 

$$\rho \times \frac{\pi}{16} y^2 dy = \frac{\rho\pi}{16} y^2 dy$$

- weight of the slice 

$$g \times \frac{\rho\pi}{16} y^2 \, dy$$

$$= \frac{g\rho\pi}{16} y^2 \, dy$$

- work to lift the slice
to the top

$$(6-y) \times \frac{g\rho\pi}{16} y^2 \, dy$$

$$= \frac{g\rho\pi}{16} (6-y) y^2 \, dy.$$

- Total work

$$\int_0^6 \frac{g\rho\pi}{16} (6-y) y^2 \, dy$$

$$= \frac{g\rho\pi}{16} \int_0^6 (6-y) y^2 \, dy$$

$$= \frac{g\rho\pi}{16}\int_0^6 (6y^2 - y^3)\,dy.$$

(47)

$$= \frac{g\rho\pi}{16}\left[6\cdot\frac{y^3}{3} - \frac{y^4}{4}\right]_0^6$$

$$= \frac{9.8 \times 1000 \times \pi}{16} \times \frac{1}{12}6^4$$

$$= 66150\pi. \quad J$$

b. No.

48

$$\frac{1}{2}\int_0^6 \frac{g\rho\pi}{16}(6-y)\,y^2\,dy$$

$\neq$ (actually $>$)

$$\int_0^3 \frac{g\rho\pi}{16}(6-y)\,y^2\,dy$$

6.

8.2.9.

$$\int x \cos 5x \, dx = \int u \, dv$$

$$\left(\begin{array}{ll} u = x & v = \frac{1}{5} \sin 5x \\ du = dx & dv = \cos 5x \, dx \end{array} \right)$$

$$= uv - \int v \, du$$

$$= x \cdot \frac{1}{5} \sin 5x - \int \frac{1}{5} \sin 5x \, dx$$

$$= x \cdot \frac{1}{5} \sin 5x - \frac{1}{5} \left(-\frac{1}{5} \cos 5x \right) + C$$

$$= \frac{1}{5} x \sin 5x + \frac{1}{25} \cos 5x + C$$

8.2.10

(50)

$$\int x \sin 2x \, dx = \int u \, dv$$

$$\left(\begin{array}{ll} u = x & v = -\frac{1}{2}\cos 2x \\ du = dx & dv = \sin 2x \, dx \end{array} \right)$$

$$= uv - \int v \, du$$

$$= x\left(-\frac{1}{2}\cos 2x\right) - \int \left(-\frac{1}{2}\cos 2x\right) dx$$

$$= x\left(-\frac{1}{2}\cos 2x\right) + \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + C$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C.$$

8.2.12.

$$\int 2x\, e^{3x} dx = 2 \int x e^{3x}\, dx$$

$$\int x e^{3x} dx = \int u\, dv$$

$$\left(\begin{array}{ll} u = x & v = \frac{1}{3} e^{3x} \\ du = dx & dv = e^{3x} dx \end{array} \right)$$

$$= uv - \int v\, du$$

$$= x \cdot \left(\frac{1}{3} e^{3x} \right) - \int \frac{1}{3} e^{3x} dx$$

$$= x \left(\frac{1}{3} e^{3x} \right) - \frac{1}{3} \left(\frac{1}{3} e^{3x} \right) + C$$

$$= \frac{1}{3} x\, e^{3x} - \frac{1}{9} e^{3x} + C$$

Final answer

$$\int 2x e^{3x} dx$$

$$= \frac{2}{3} x e^{3x} - \frac{2}{9} e^{3x} + C.$$

8.2.13

$$\int x \ln(10x)\, dx = \int u\, dv$$

$$\left(\begin{array}{ll} u = \ln(10x) & v = \frac{1}{2}x^2 \\ du = \frac{1}{x}\, dx & dv = x\, dx \end{array} \right)$$

$$= uv - \int v\, du$$

$$= \ln(10x) \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{2}x^2 \ln(10x) - \frac{1}{2}\int x\, dx$$

$$= \frac{1}{2}x^2 \cdot \ln(10x) - \frac{1}{2} \cdot \frac{x^2}{2} + C$$

$$= \frac{1}{2}x^2 \cdot \ln(10x) - \frac{1}{4}x^2 + C$$

7.

8.3.9

$$\int \cos^3 x \, dx$$

$$= \int \cos^2 x \cdot \cos x \, dx$$

$$\left(\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right)$$

$$= \int (1 - \sin^2 x) \cos x \, dx$$

$$= \int (1 - u^2) \, du$$

$$= u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

8.3.11

$$\int \sin^2(3x)\,dx$$

$$= \int \frac{1-\cos(6x)}{2}\,dx$$

$$= \frac{1}{2}\int \{1-\cos(6x)\}\,dx$$

$$= \frac{1}{2}\left\{x - \frac{1}{6}\sin(6x)\right\} + C$$

$$= \frac{1}{2}x - \frac{1}{12}\sin(6x) + C.$$

8.3.13

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$$\int \sin^5 x \, dx$$

$$= \int \sin^4 x \cdot \sin x \, dx$$

$$\left(\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right)$$

$$= \int (1-\cos^2 x)^2 \cdot \sin x \, dx$$

$$= \int (1-u^2)^2 (-du)$$

$$= -\int (u^4 - 2u^2 + 1) \, du$$

$$= -\frac{u^5}{5} + 2 \cdot \frac{u^3}{3} - u + C$$

$$= -\frac{\cos^5 x}{5} + \frac{2}{3}\cos^3 x - \cos x + C$$

8.3.14

$$\int \cos^3(20x)\,dx$$

$$= \int \cos^2(20x)\cos(20x)\,dx$$

$$\left(\begin{array}{l} u = \sin(20x) \\ du = 20\cos(20x)\,dx \end{array} \right)$$

$$= \int \{1 - \sin^2(20x)\}\cos(20x)\,dx$$

$$= \int (1 - u^2)\frac{1}{20}\,du$$

$$= \frac{1}{20}\int (1 - u^2)\,du$$

$$= \frac{1}{20}\left(u - \frac{u^3}{3}\right) + C$$

$$= \frac{1}{20}\sin(20x) - \frac{1}{60}\sin^3(20x) + C$$

8.3.15

(58)

$$\int \sin^3 x \cos^2 x \, dx$$

$$= \int \sin^2 x \cos^2 x \sin x \, dx$$

$$\left(\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right)$$

$$= \int (1 - \cos^2 x) \cos^2 x \cdot \sin x \, dx$$

$$= \int (1 - u^2) u^2 (-du)$$

$$= \int (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C$$

8.3.27

59

$$\int \tan^2 x \, dx$$

$$= \int (\sec^2 x - 1) \, dx$$

$$= \tan x - x + C.$$

8.3.34

60

$$\int \tan^9 x \sec^4 x \, dx$$

$$= \int \tan^9 x \sec^2 x \sec^2 x \, dx$$

$$\left(\begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array} \right)$$

$$= \int \tan^9 x \left(1 + \tan^2 x\right) \sec^2 dx$$

$$= \int u^9 \left(1 + u^2\right) du$$

$$= \int \left(u^9 + u^{11}\right) du$$

$$= \frac{u^{10}}{10} + \frac{u^{12}}{12} + C$$

$$= \frac{1}{10} \tan^{10} x + \frac{1}{12} \tan^{12} x + C$$

8.3.35

61

$$\int \tan x \sec^3 x \, dx$$

$$= \int \sec^2 x \ \tan x \sec x \, dx$$

$$\left(\begin{array}{l} u = \sec x \\ du = \tan x \sec x \, dx \end{array} \right)$$

$$= \int u^2 \, du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{1}{3} \sec^3 x + C$$

8

8.4.11

$$\int_{1/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$\left(\begin{array}{cccc} x & x = \sin\theta & \theta & dx = \cos\theta\,d\theta \\ \sqrt{3}/2 & & \pi/3 & \\ 1/2 & & \pi/6 & \end{array}\right)$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin^2\theta}{\cancel{\cos\theta}}\cancel{\cos\theta}\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \sin^2\theta\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1 - \cos(2\theta)}{2} \, d\theta$$

$$= \frac{1}{2} \left[\theta - \frac{1}{2} \sin(2\theta) \right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2} \left[\left\{ \pi/3 - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right\} - \left\{ \pi/6 - \frac{1}{2} \frac{\sqrt{3}}{2} \right\} \right]$$

$$= \frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}$$

8.4.16

64

$$\int \frac{x^2}{(25+x^2)^2}\, dx$$

$$\left(\begin{array}{l} x = 5\tan\theta \\ dx = 5\sec^2\theta\, d\theta \end{array} \right)$$

$$= \int \frac{(5\tan\theta)^2}{\{25 + (5\tan\theta)^2\}^2}\, 5\sec^2\theta\, d\theta$$

$$= \int \frac{25\tan^2\theta}{(25\sec^2\theta)^2}\, 5\sec^2\theta\, dx$$

$$= \frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{5}\int \sin^2\theta\, d\theta$$

$$= \frac{1}{5} \int \frac{1 - \cos(2\theta)}{2} \, d\theta$$

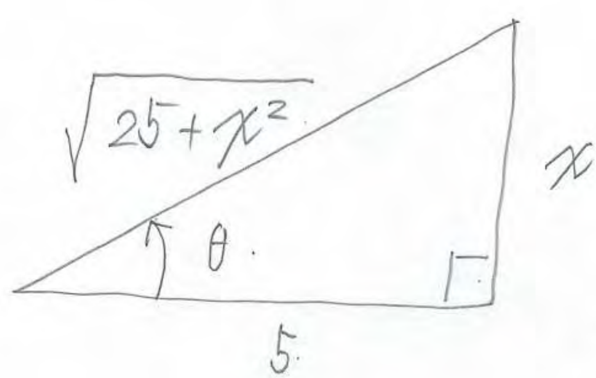
$$= \frac{1}{10} \int \{ 1 - \cos(2\theta) \} \, d\theta$$

$$= \frac{1}{10} \{ \theta - \frac{1}{2} \underbrace{\sin(2\theta)}_{\parallel} \} + C$$

$$= \frac{1}{10} \{ \theta - \frac{1}{2} \overbrace{2 \sin\theta \cos\theta} \} + C$$

$$= \frac{1}{10} \{ \theta - \sin\theta \cos\theta \} + C$$

$$x = 5 \tan\theta \qquad \frac{x}{5} = \tan\theta$$



$$= \frac{1}{10}\left\{ \tan^{-1}\left(\frac{x}{5}\right) - \frac{x}{\sqrt{25+x^2}} \cdot \frac{5}{\sqrt{25+x^2}} \right\} + C$$

$$= \frac{1}{10}\left\{ \tan^{-1}\left(\frac{x}{5}\right) - \frac{5x}{25+x^2} \right\} + C$$

8.4.19

67

$$\int \frac{dx}{\sqrt{x^2-81}}, \quad x > 9$$

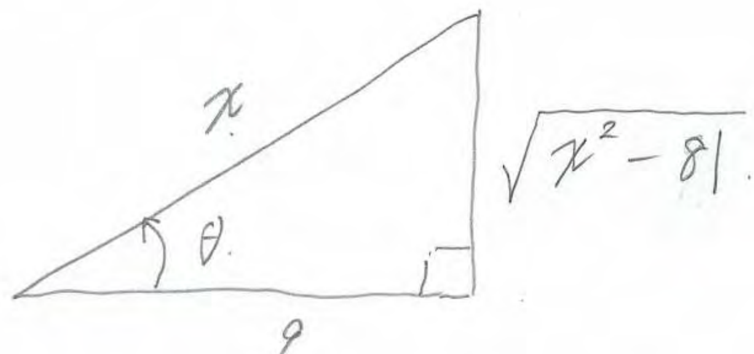
$$\left(\begin{aligned} x &= 9\sec\theta, \quad 0 < \theta < \tfrac{\pi}{2} \\ dx &= 9\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-81} &= \sqrt{(9\sec\theta)^2-81} \\ &= \sqrt{81(\sec^2\theta-1)} \\ &= \sqrt{81\tan^2\theta} \\ &= 9\tan\theta \\ &\left(\tan\theta > 0 \Leftarrow 0 < \theta < \tfrac{\pi}{2}\right) \end{aligned} \right)$$

$$= \int \frac{9\sec\theta\tan\theta\, d\theta}{9\tan\theta}$$

$$= \int \sec\theta \, d\theta$$

$$= \ln \left| \sec\theta + \tan\theta \right| + C$$

$$\left(x = 9\sec\theta \qquad \frac{x}{9} = \sec\theta \right.$$



$$\left. \right)$$

$$= \ln \left| \frac{x}{9} + \frac{\sqrt{x^2-81}}{9} \right| + C$$

$$= \ln \left(x + \sqrt{x^2 - 81} \right) + C$$

$\Big(\leftarrow x > 9$

& replace "old C" with

"new C" = "old C" $- \ln 9 \Big)$

8.4.22

$$\int \frac{dt}{t^2\sqrt{9-t^2}}$$

$$\left(\begin{aligned} t &= 3\sin\theta \\ dt &= 3\cos\theta\, d\theta \\ \sqrt{9-t^2} &= \sqrt{9-(3\sin\theta)^2} \\ &= \sqrt{9(1-\sin^2\theta)} \\ &= \sqrt{9\cos^2\theta} = 3\cos\theta \end{aligned}\right)$$

$$= \int \frac{\cancel{3\cos\theta}\, d\theta}{(3\sin\theta)^2\, \cancel{3\cos\theta}}$$

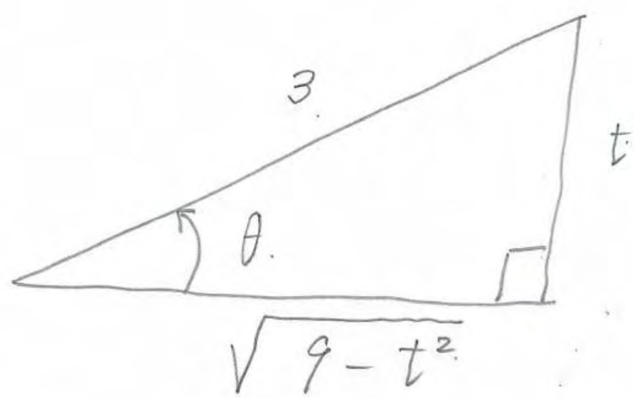
$$= \frac{1}{9}\int \frac{1}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{9}\int \csc^2\theta\, d\theta$$

$$= \frac{1}{9}\left(-\cot\theta\right) + C$$

$$= -\frac{1}{9}\cot\theta + C$$

$$t = 3\sin\theta \qquad \frac{t}{3} = \sin\theta$$



$$= -\frac{1}{9}\cdot\frac{\sqrt{9-t^2}}{t} + C$$

$$\int \frac{\sqrt{9-x^2}}{x}\,dx$$

$$\left(\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos\theta\,d\theta \\ \sqrt{9-x^2} &= 3\cos\theta \end{aligned}\right)$$

$$= \int \frac{3\cos\theta}{3\sin\theta}\,3\cos\theta\,d\theta$$

$$= 3\int \frac{\cos^2\theta}{\sin\theta}\,d\theta$$

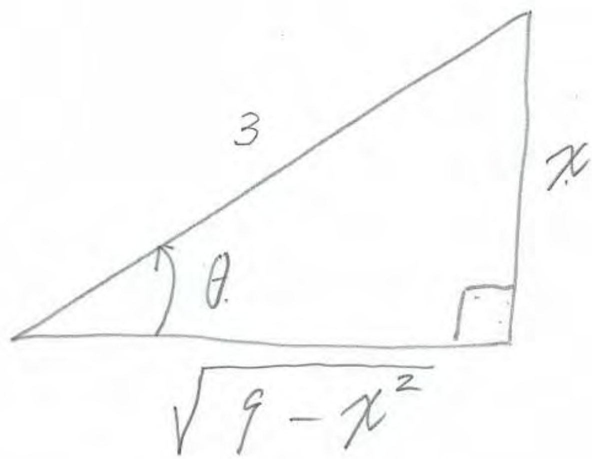
$$= 3\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$= 3\int (\csc\theta - \sin\theta)\,d\theta$$

$$= 3\left\{ -\ln|\csc\theta + \cot\theta| + \cos\theta \right\} + C$$

$$= -3\ln|\csc\theta + \cot\theta| + 3\cos\theta + C$$

$$x = 3\sin\theta \qquad \frac{x}{3} = \sin\theta$$



$$= -3\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| + 3\cdot\frac{\sqrt{9-x^2}}{3} + C$$

$$= -3\ln\left|\frac{3+\sqrt{9-x^2}}{x}\right| + \sqrt{9-x^2} + C$$

8.4.26

$$\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x}\,dx$$

x	$x = \sec\theta$	θ	$dx = \sec\theta\tan\theta\,d\theta$
2		$\pi/3$	
$\sqrt{2}$		$\pi/4$	

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1}$$

$$= \sqrt{\tan^2\theta} = \tan\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{\tan\theta}{\cancel{\sec\theta}}\,\cancel{\sec\theta}\tan\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \tan^2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} (\sec^2\theta - 1)\, d\theta$$

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$$= \left[\tan\theta - \theta \right]_{\pi/4}^{\pi/3}$$

$$= \left[\left(\sqrt{3} - \pi/3\right) - \left(1 - \pi/4\right) \right]$$

$$= \sqrt{3} - 1 + \frac{\pi}{12}.$$

9.

8.5.24

$$\int \frac{8}{(x-2)(x+6)}\,dx$$

partial fractions

$$\frac{8}{(x-2)(x+6)} = \frac{A}{x-2} + \frac{B}{x+6}$$

$\times (x-2)(x+6)$

$$8 = A(x+6) + B(x-2)$$

$x = 2 \quad 8 = A(2+6) \qquad A = 1$

$x = -6 \quad 8 = B(-6-2) \qquad B = -1$

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$$\int \frac{8}{(x-2)(x+6)}\,dx$$

$$= \int \left\{ \frac{1}{x-2} + \frac{-1}{x+6} \right\} dx$$

$$= \ln|x-2| - \ln|x+6| + C.$$

(or

$$= \ln\left|\frac{x-2}{x+6}\right| + C.$$)

8.5.26

$$\int_0^1 \frac{dt}{t^2-9}$$

partial fractions

$$\frac{1}{t^2-9} = \frac{1}{(t-3)(t+3)} = \frac{A}{t-3} + \frac{B}{t+3}$$

$\xrightarrow{\times (t-3)(t+3)}$

$$1 = A(t+3) + B(t-3)$$

$t = 3$

$$1 = A(3+3) \qquad A = 1/6$$

$t = -3 \qquad 1 = B(-3-3) \qquad B = -1/6$

$$= \int_0^1 \left\{ \frac{1/6}{t-3} + \frac{-1/6}{t+3} \right\} dt$$

$$= \frac{1}{6}\int_0^1 \left\{ \frac{1}{t-3} - \frac{1}{t+3} \right\} dt$$ 78

$$= \frac{1}{6} \left[\ln|t-3| - \ln|t+3| \right]_0^1$$

$$= \frac{1}{6} \left[(\ln 2 - \ln 4) - (\ln 3 - \ln 3) \right]$$

$$= \frac{1}{6} \left[\ln 2 - 2\ln 2 \right]$$

$$= -\frac{\ln 2}{6}$$

8.5.29.

$$\int_{-1}^{2} \frac{5x}{x^2 - x - 6}\, dx$$

partial fractions

$$\frac{5x}{x^2-x-6} = \frac{5x}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$\xrightarrow{\times (x-3)(x+2)}$$

$$5x = A(x+2) + B(x-3)$$

$x = 3$

$$5 \cdot 3 = A(3+2) \qquad A = 3$$

$x = -2$

$$5(-2) = B(-2-3) \qquad B = 2$$

$$= \int_{-1}^{2} \left\{ \frac{3}{x-3} + \frac{2}{x+2} \right\} dx$$

$$= \left[3 \ln|x-3| + 2 \ln|x+2| \right]_{-1}^{2}$$

$$= \left[(3 \underbrace{\ln 1}_{0} + 2 \ln 4) - (3 \ln 4 + 2 \underbrace{\ln 1}_{0}) \right]$$

$$= -\ln 4$$

$$\left(\overset{\text{or}}{=} -2 \ln 2 \right)$$

10.

8.9.12

$$\int_{-\infty}^{-1} \frac{dx}{\sqrt[3]{x}}$$

$$= \lim_{b \to -\infty} \int_{b}^{-1} \frac{dx}{\sqrt[3]{x}}$$

$$= \lim_{b \to -\infty} \left[\frac{3}{2} x^{\frac{2}{3}} \right]_{b}^{-1}$$

$$= \lim_{b \to -\infty} \frac{3}{2} \left[1 - \left(\sqrt[3]{b} \right)^{2} \right] = -\infty$$

$$\int_{-\infty}^{-1} \frac{dx}{\sqrt[3]{x}} \quad \text{diverges}$$

8. 9. 18

$$\int_2^\infty \frac{dx}{(x+2)^2}$$

$$= \lim_{b\to\infty} \int_2^b \frac{dx}{(x+2)^2}$$

$$= \lim_{b\to\infty} \left[-\frac{1}{x+2} \right]_2^b$$

$$= \lim_{b\to\infty} \left[\left(-\underbrace{\frac{1}{b+2}}_{\downarrow\, 0} \right) - \left(-\frac{1}{2+2} \right) \right]$$

$$= \frac{1}{4}$$

$\displaystyle\int_2^\infty \frac{dx}{(x+2)^2}$ converges to $\frac{1}{4}$.

8.9.27

$$\int_{-\infty}^{\infty} x e^{-x^2} dx$$

$$= \int_{-\infty}^{0} x e^{-x^2} dx + \int_{0}^{\infty} x e^{-x^2} dx$$

$$\int_{-\infty}^{0} x e^{-x^2} dx = \lim_{b \to \infty} \int_{b}^{0} x e^{-x^2} dx$$

$$= \lim_{b \to -\infty} \left[-\frac{1}{2} e^{-x^2} \right]_{b}^{0}$$

$$= \lim_{b \to \infty} -\frac{1}{2} [\underbrace{e^{-0}}_{= 1} - \underbrace{e^{-b^2}}_{\to 0}] = -\frac{1}{2}$$

converges

$$\int_{0}^{\infty} x e^{-x^2} dx = \lim_{b \to \infty} \int_{0}^{b} x e^{-x^2} dx$$

$$= \lim_{b\to\infty} \left[-\frac{1}{2} e^{-x^2}\right]_0^b$$

$$= \lim_{b\to\infty} -\frac{1}{2}\left[\underset{\downarrow \atop 0}{e^{-b^2}} - \underbrace{e^{-0}}_{\overset{\shortparallel}{1}}\right] = \frac{1}{2}$$

<u>Conclusion</u>

$$\int_{-\infty}^{\infty} x e^{-x^2} dx$$

$$= \underbrace{\int_{-\infty}^{0} x e^{-x^2} dx}_{\substack{= \\ -\frac{1}{2} \\ \text{converges}}} + \underbrace{\int_{0}^{\infty} x e^{-x^2} dx}_{\substack{= \\ \frac{1}{2} \\ \text{converges}}}$$

$= 0$ converges.

Warning:

The following is a WRONG argument:

$$\int_{-\infty}^{\infty} x e^{-x^2}\, dx = \lim_{b \to \infty} \underbrace{\int_{-b}^{b} x e^{-x^2}\, dx}_{\substack{\parallel \\ 0}}$$

since $f(x) = x e^{-x^2}$
is an odd function

$$= 0.$$

Example

$\int_{-\infty}^{\infty} x\, dx$ diverges, even though

$$\lim_{b \to \infty} \int_{-b}^{b} x\, dx = 0$$

11.

10.4.9

$$\sum_{k=0}^{\infty} \frac{k}{2k+1} = \sum_{k=0}^{\infty} a_k$$

where

$$a_k = \frac{k}{2k+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2} \neq 0$$

$$\xrightarrow{\text{D.T.}} \sum_{k=0}^{\infty} a_k \text{ diverges.}$$

10.4.10

$$\sum_{k=1}^{\infty} \frac{k}{k^2+1} = \sum_{k=1}^{\infty} a_k$$

where

$$a_k = \frac{k}{k^2+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k^2+1} = 0$$

$\longrightarrow$ Divergence Test is inconclusive

Note: By the Limit Comparison Test with $b_k = \frac{k}{k^2} = \frac{1}{k}$, the series $\sum a_k$ actually diverges.

10.4.11

(88)

$$\sum_{k=0}^{\infty} \frac{1}{1000+k} = \sum_{k=0}^{\infty} a_k$$

where

$$a_k = \frac{1}{1000+k}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{1000+k} = 0.$$

$\longrightarrow$

Divergence Test is inconclusive

Note: By the Limit Comparison Test with $b_k = \frac{1}{k}$ $(k \geq 1)$, the series $\sum_{k=0}^{\infty} a_k$ actually diverges.

10.4.12.

$$\sum_{k=1}^{\infty} \frac{k^3}{k^3+1} = \sum_{k=1}^{\infty} a_k \qquad 89$$

where

$$a_k = \frac{k^3}{k^3+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^3}{k^3+1} = 1 \neq 0$$

$$\xrightarrow{\text{D.T.}} \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

10.4.13

$$\sum_{k=2}^{\infty} \frac{k}{\ln k} = \sum_{k=2}^{\infty} a_k$$

where

$$a_k = \frac{k}{\ln k}$$

$$\lim_{k \to \infty} \frac{k}{\ln k} = \infty \neq 0$$

$$\xrightarrow{\text{D.T.}} \sum_{k=2}^{\infty} a_k \text{ diverges}$$

Note:

$$\lim_{x \to \infty} \frac{x}{\ln x} \quad \text{formally of the form } \left(\frac{\infty}{\infty}\right)$$

$\underline{\underline{\text{L.H.}}}$

$$\lim_{x \to \infty} \frac{1}{1/x} = \lim_{x \to \infty} x = \infty.$$

12.

$\underline{10.4.21}$

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$$\sum_{k=1}^{\infty} k e^{-2k^2} \qquad a_k = k e^{-2k^2}$$

Consider

$$f(x) = x e^{-2x^2} \text{ over } [1, \infty)$$

(i) continuous ✓

(ii) positive ✓

(iii) decreasing

since

$$f'(x) = 1 \cdot e^{-2x^2} + x(-4x) e^{-2x^2}$$

$$= (1 - 4x^2) e^{-2x^2} < 0$$

$$\text{on } [1, \infty)$$

&

$$f(k) = a_k$$

Integral Test $\longrightarrow$ $\sum_{k=1}^{\infty} a_k$ & $\int^{\infty} f(x)\, dx$ share the same destiny.

We compute

$$\int_1^\infty f(x)\,dx$$

$$= \int_1^\infty x e^{-2x^2} dx$$

$$= \lim_{b\to\infty} \int_1^b x e^{-2x^2} dx$$

$$= \lim_{b\to\infty} \left[-\frac{1}{4} e^{-2x^2} \right]_1^b$$

$$= \lim_{b\to\infty} -\frac{1}{4} \left[\underset{\substack{\downarrow \\ 0}}{e^{-2b^2}} - e^{-2\cdot 1} \right] = \frac{1}{4} e^{-2}$$

converges

<u>Conclusion</u>

$\sum_{k=1}^{\infty} a_k$ converges.

10.4.22.

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} \qquad a_k = \frac{1}{k(\ln k)^2}$$

Consider

$$f(x) = \frac{1}{x(\ln x)^2} \quad \text{over } [2, \infty)$$

(i) continuous ✓

(ii) positive ✓

(iii) decreasing ✓

since

$$g(x) = x(\ln x)^2 \text{ increasing}$$

&

$$f(k) = a_k$$

$\xrightarrow{\text{Integral Test}}$

$$\sum_{k=2}^{\infty} a_k \text{ \& } \int_2^{\infty} f(x)\,dx$$

share the same destiny

We compute

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x(\ln x)^2}\,dx$$

$$= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx$$

$$\left(\begin{array}{ll} x & u = \ln x \\ b & \ln b \\ 2 & \ln 2 \end{array} \quad , \quad du = \frac{1}{x}\,dx \right)$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^2}\,du$$

$$= \lim_{b\to\infty} \left[-\frac{1}{u} \right]_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} \left[\left(\underbrace{-\frac{1}{\ln b}}_{\to\, 0}\right) - \left(-\frac{1}{\ln 2}\right) \right] = \frac{1}{\ln 2}$$

$$\int_2^\infty f(x)\,dx = \frac{1}{\ln 2}$$

converges.

Conclusion

$\sum_{k=2}^\infty a_k$ converges.

10.4.33

$$\sum_{k=1}^{\infty} \frac{k}{e^k} \qquad a_k = \frac{k}{e^k} = k e^{-k}$$

Consider

$$f(x) = x e^{-x} \text{ over } [1, \infty)$$

(i) continuous ✓

(ii) positive ✓

(iii) decreasing

since

$$f'(x) = 1 \cdot e^{-x} + x(-e^{-x})$$

$$= (1-x) e^{-x} < 0$$

& over $(1, \infty)$

$$f(k) = a_k$$

$\xrightarrow{\text{Integral Test}}$

$$\sum_{k=1}^{\infty} a_k \ \& \ \int_1^{\infty} f(x)\, dx$$

share the same destiny.

We compute

$$\int_1^\infty f(x)\,dx = \int_1^\infty x e^{-x}\,dx$$

$$= \lim_{b\to\infty} \int_1^b x e^{-x}\,dx$$

$$\begin{pmatrix} u = x & v = -e^{-x} \\ du = dx & dv = e^{-x}dx \end{pmatrix}$$

$$= \lim_{b\to\infty} \int_1^b u\,dv$$

$$= \lim_{b\to\infty} \left\{ [uv]_1^b - \int_1^b v\,du \right\}$$

$$= \lim_{b\to\infty} \left\{ [x(-e^{-x})]_1^b - \int_1^b (-e^{-x})\,dx \right\}$$

$$= \lim_{b\to\infty} \left\{ -[\underbrace{b e^{-b}}_{\downarrow\, 0} - 1\cdot e^{-1}] + \underbrace{[-e^{-x}]_1^b}_{(-e^{-b})-(-e^{-1})} \right\}$$

$$0 \leftarrow (-e^{-b})$$

$= 2e^{-1}$

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converges.

(Note: $\lim_{x \to \infty} x e^{-x} = \lim_{x \to \infty} \frac{x}{e^x} \quad \left(\frac{\infty}{\infty}\right)$

$\overset{L.H.}{=} \lim_{x \to \infty} \frac{1}{e^x} = 0$)

Conclusion

$\sum_{k=1}^{\infty} a_k$ converges.

13.

10.6.25

$$\sum_{k=1}^{\infty} (-1)^{k+1} k^{\frac{1}{k}}$$

Set $b_k = k^{\frac{1}{k}}$, $a_k = (-1)^{k+1} b_k$

Claim $\lim_{k\to\infty} k^{\frac{1}{k}} = 1$

$\because)$ Set $y = x^{\frac{1}{x}}$

$$\ln y = \frac{1}{x} \ln x$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{L.H.}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$\longrightarrow$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\overset{\to 0}{\ln y}} = e^0 = 1.$$

$\longrightarrow$ $\lim_{k\to\infty} b_k = 1$

$\longrightarrow$ $\lim_{k\to\infty} a_k$ D.N.E.

P.T. $\longrightarrow$ $\sum_{k=1}^{\infty} a_k$ diverges.

10.6.26

$$\sum_{k=1}^{\infty} (-1)^k k \sin\left(\frac{1}{k}\right)$$

Set $b_k = k \sin\left(\frac{1}{k}\right)$

$a_k = (-1)^k b_k$

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} k \sin\left(\frac{1}{k}\right)$$

$$= \lim_{k\to\infty} \frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} = 1$$

$\rightarrow$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (-1)^k b_k \quad \text{D.N.E.}$$

$\xrightarrow{\text{D.T.}}$ $\sum_{k=1}^{\infty} a_k$ diverges.

14.

10.8.85

$$\frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + \frac{1}{6\cdot 7} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)}$$

Set $a_k = \dfrac{1}{2k(2k+1)}$

$\longrightarrow$

$$0 \leq a_k = \frac{1}{2k(2k+1)} \leq \frac{1}{2k\cdot 2k} = \frac{1}{4}\frac{1}{k^2} = b_k$$

$$\sum_{k=1}^{\infty} a_k \xleftarrow{\text{C.T.}} \sum b_k = \frac{1}{4}\sum \frac{1}{k^2}$$

converges. p-series with $p > 1$ converges

10.8.88

$$\sum_{k=1}^{\infty} \left(\sqrt{k^4+1} - k^2 \right)$$

$$\begin{aligned}
a_k &= \sqrt{k^4+1} - k^2 \\
&= \frac{\left(\sqrt{k^4+1} - k^2\right)\left(\sqrt{k^4+1} + k^2\right)}{\sqrt{k^4+1} + k^2} \\
&= \frac{\left(\sqrt{k^4+1}\right)^2 - \left(k^2\right)^2}{\sqrt{k^4+1} + k^2} \\
&= \frac{1}{\sqrt{k^4+1} + k^2}
\end{aligned}$$

$\longrightarrow$

$$0 \leq a_k = \frac{1}{\sqrt{k^4+1} + k^2} \leq \frac{1}{\sqrt{k^4} + k^2}$$

$$\frac{1}{k^2+k^2} = \frac{1}{2k^2} = b_k$$

$\sum a_k$ converges. $\xleftarrow{\text{C.T.}}$ $\sum b_k = \frac{1}{2} \sum \frac{1}{k^2}$ converges p-series with $p = 2 > 1$.

10.8.93

(103)

$$\sum_{k=1}^{\infty} \frac{1}{2^{\ln k} + 2}$$

$$a_k = \frac{1}{2^{\ln k} + 2}$$

Observe

$$\begin{cases} \ln k \geq 1 & \text{for } k \geq 1 \\ 2 < e \end{cases}$$

$$\longrightarrow \quad 2^{\ln k} < e^{\ln k}$$

$$\longrightarrow \quad 2^{\ln k} + 2 < e^{\ln k} + 2 = k + 2$$

$$\longrightarrow \quad b_k = \frac{1}{k+2} < \frac{1}{2^{\ln k} + 2}$$

$$\longrightarrow$$

$$0 \leq b_k = \frac{1}{k+2} < \frac{1}{2^{\ln k}+2} = a_k$$

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k+2} \quad \text{diverges}$$

$$= \sum_{k=3}^{\infty} \frac{1}{k} \quad \text{harmonic ser}$$

$$\xrightarrow{\text{C.T.}} \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

15.

11.2.66

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$$

interval of conv.

$$I = (-1, 1]$$

i.e.

converges

$$\Longleftrightarrow \quad x \in (-1, 1]$$

$$\ln x = \ln(1+(x-1))$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k$$

converges

$$\Longleftrightarrow$$

$$x-1 \in (-1, 1]$$

$$\Longleftrightarrow$$

$$x \in (0, 2]$$

i.e.

$$I = (0, 2]$$

11 . 2 . 70

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$= \sum_{k=0}^{\infty} x^k$$

converges.

$\Longleftrightarrow$

$x \in (-1, 1)$

i.e. $I = (-1, 1)$

$\longrightarrow$

$$\frac{1}{1-x} - 1 = x + x^2 + \cdots$$

$$= \sum_{k=1}^{\infty} x^k$$

$\|$

$$\frac{1-(1-x)}{1-x}$$

converges.

$\Longleftrightarrow$

$x \in (-1, 1)$

$\|$

$$\frac{x}{1-x}$$

i.e. $I = (-1, 1)$

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{3^{2k}}$$

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$$= \sum_{k=1}^{\infty} \left(\frac{x-2}{3^2}\right)^k$$

$$= \frac{\left(\frac{x-2}{9}\right)}{1-\left(\frac{x-2}{9}\right)} = \frac{x-2}{9-(x-2)}$$

$$= \frac{x-2}{11-x}$$

converges

$\iff$

$$\frac{x-2}{9} \in (-1, 1)$$

$\iff$

$$x-2 \in (-9, 9)$$

$\iff$

$$x \in (-7, 11).$$

i.e.

$$I = (-7, 11)$$

$\lrcorner$