CHAPTER
1

First-Order Differential Equations

Among all of the mathematical disciplines the theory of differential equations is the most important. It furnishes the explanation of all those elementary manifestations of nature which involve time. — Sophus Lie

1.1 How Differential Equations Arise

In this section we will introduce the idea of a differential equation through the mathematical formulation of a variety of problems. We then use these problems throughout the chapter to illustrate the applicability of the techniques introduced.

Newton's Second Law of Motion

Newton's second law of motion states that, for an object of constant mass m, the sum of the applied forces acting on the object is equal to the mass of the object multiplied by the acceleration of the object. If the object is moving in one dimension under the influence of a force F, then the mathematical statement of this law is

$$m\frac{dv}{dt} = F, \tag{1.1.1}$$

where $v(t)$ denotes the velocity of the object at time t. We let $y(t)$ denote the displacement of the object at time t. Then, using the fact that velocity and displacement are related via

$$v = \frac{dy}{dt},$$

we can write (1.1.1) as

$$m\frac{d^2y}{dt^2} = F. \tag{1.1.2}$$

This is an example of a **differential equation**, so called because it involves *derivatives* of the unknown function $y(t)$.

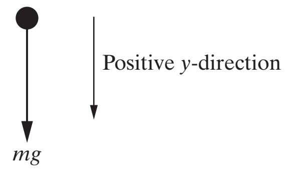

Figure 1.1.1: Object falling under the influence of gravity.

Gravitational Force: As a specific example, consider the case of an object falling freely under the influence of gravity (see Figure 1.1.1). In this case the only force acting on the object is $F = mg$, where g denotes the (constant) acceleration due to gravity. Choosing the positive y-direction as downward, it follows from Equation (1.1.2) that the motion of the object is governed by the differential equation

$$m\frac{d^2y}{dt^2} = mg, \qquad (1.1.3)$$

or equivalently,

$$\frac{d^2y}{dt^2} = g.$$

Since g is a (positive) constant, we can integrate this equation to determine $y(t)$. Performing one integration yields

$$\frac{dy}{dt} = gt + c_1,$$

where c_1 is an arbitrary integration constant. Integrating once more with respect to t, we obtain

$$y(t) = \frac{1}{2}gt^2 + c_1t + c_2, \qquad (1.1.4)$$

where c_2 is a second integration constant. We see that the differential equation has an infinite number of solutions parameterized by the constants c_1 and c_2. In order to uniquely specify the motion, we must augment the differential equation with initial conditions that specify the initial position and initial velocity of the object. For example, if the object is released at $t = 0$ from $y = y_0$ with a velocity v_0, then, in addition to the differential equation, we have the initial conditions

$$y(0) = y_0, \qquad \frac{dy}{dt}(0) = v_0. \qquad (1.1.5)$$

These conditions must be imposed on the solution (1.1.4) in order to determine the values of c_1 and c_2 that correspond to the particular problem under investigation. Setting $t = 0$ in (1.1.4) and using the first initial condition from (1.1.5), we find that

$$y_0 = c_2.$$

Substituting this into Equation (1.1.4), we get

$$y(t) = \frac{1}{2}gt^2 + c_1t + y_0. \qquad (1.1.6)$$

In order to impose the second initial condition from (1.1.5), we first differentiate Equation (1.1.6) to obtain

$$\frac{dy}{dt} = gt + c_1.$$

Consequently the second initial condition in (1.1.5) requires

$$c_1 = v_0.$$

From (1.1.6), it follows that the position of the object at time t is

$$y(t) = \frac{1}{2}gt^2 + v_0 t + y_0.$$

The differential equation (1.1.3) together with the initial conditions (1.1.5) is an example of an **initial-value problem**.

Spring Force: As a second application of Newton's law of motion, consider the spring–mass system depicted in Figure 1.1.2, where, for simplicity, we are neglecting frictional and external forces. In this case, the only force acting on the mass is the restoring force (or spring force), F_s, due to the displacement of the spring from its equilibrium (unstretched) position. We use Hooke's law to model this force:

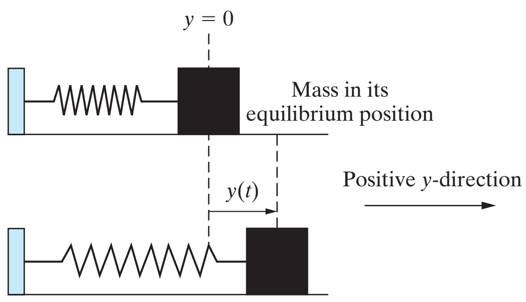

Figure 1.1.2: A simple harmonic oscillator.

Hooke's Law: The restoring force of a spring is directly proportional to the displacement of the spring from its equilibrium position and is directed toward the equilibrium position.

If $y(t)$ denotes the displacement of the spring from its equilibrium position at time t (see Figure 1.1.2), then according to Hooke's law, the restoring force is

$$F_s = -ky,$$

where k is a positive constant called the **spring constant**. Consequently, Newton's second law of motion implies that the motion of the spring–mass system is governed by the differential equation

$$m\frac{d^2y}{dt^2} = -ky,$$

which we write in the equivalent form

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \qquad (1.1.7)$$

where $\omega = \sqrt{k/m}$. At present we cannot solve this differential equation. However, we leave it as an exercise (Problem 7) to verify by direct substitution that

$$y(t) = A\cos(\omega t - \phi)$$

is a solution to the differential equation (1.1.7), where A and ϕ are constants (determined from the initial conditions for the problem). We see that the resulting motion is periodic with amplitude A. This is consistent with what we might expect physically, since no frictional forces or external forces are acting on the system. This type of motion is referred to as **simple harmonic motion**, and the physical system is called a **simple harmonic oscillator**.

Newton's Law of Cooling

We now build a mathematical model describing the cooling (or heating) of an object. Suppose that we bring an object into a room. If the temperature of the object is hotter than that of the room, then the object will begin to cool. Further, we might expect that the major factor governing the rate at which the object cools is the temperature difference between it and the room.

Newton's Law of Cooling: The rate of change of temperature of an object is proportional to the temperature difference between the object and its surrounding medium.

To formulate this law mathematically, we let $T(t)$ denote the temperature of the object at time t, and let $T_m(t)$ denote the temperature of the surrounding medium. Newton's law of cooling can then be expressed as the differential equation

$$\frac{dT}{dt} = -k(T - T_m), \tag{1.1.8}$$

where k is a constant. The minus sign in front of the constant k is traditional. It ensures that k will always be positive.[1] After we study Section 1.4, it will be easy to show that, when T_m is constant, the solution to this differential equation is

$$T(t) = T_m + ce^{-kt}, \tag{1.1.9}$$

where c is a constant (see also Problem 12). Newton's law of cooling therefore predicts that as t approaches infinity ($t \to \infty$), the temperature of the object approaches that of the surrounding medium ($T \to T_m$). This is certainly consistent with our everyday experience (see Figure 1.1.3).

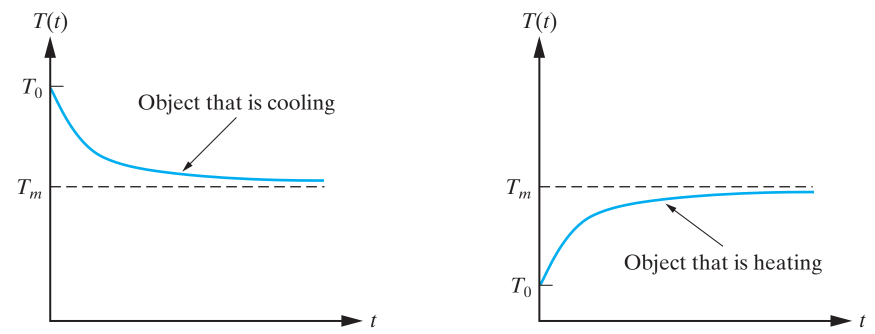

Figure 1.1.3: According to Newton's law of cooling, the temperature of an object approaches room temperature exponentially.

The Orthogonal Trajectory Problem

Next we consider a geometric problem that has many interesting and important applications. Suppose

$$F(x, y, c) = 0 \tag{1.1.10}$$

[1] If $T > T_m$, then the object will cool, so that $dT/dt < 0$. Hence, from Equation (1.1.8), k must be positive. Similarly, if $T < T_m$, then $dT/dt > 0$, and once more Equation (1.1.8) implies that k must be positive.

defines a family of curves in the xy-plane, where the constant c labels the different curves. For instance, the equation

$$x^2 + y^2 - c = 0$$

describes a family of concentric circles with center at the origin, whereas

$$-x^2 + y - c = 0$$

describes a family of parabolas that are vertical shifts of the standard parabola $y = x^2$.

We assume that every curve in the family $F(x, y, c) = 0$ has a well-defined tangent line at each point. Associated with this family is a second family of curves, say,

$$G(x, y, k) = 0, \tag{1.1.11}$$

with the property that whenever a curve from the family (1.1.10) intersects a curve from the family (1.1.11), it does so at right angles.[2] We say that the curves in the family (1.1.11) are **orthogonal trajectories** of the family (1.1.10), and vice versa. For example, from elementary geometry, it follows that the lines $y = kx$ in the family $G(x, y, k) = y - kx = 0$ are orthogonal trajectories of the family of concentric circles $x^2 + y^2 = c^2$. (See Figure 1.1.4.)

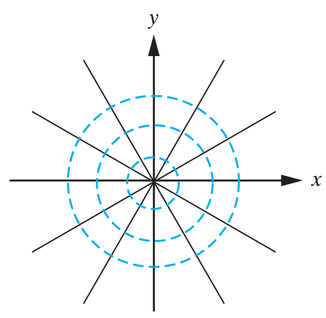

Figure 1.1.4: The family of curves $x^2 + y^2 = c^2$ and the orthogonal trajectories $y = kx$.

Orthogonal trajectories arise in various applications. For example, a family of curves and its orthogonal trajectories can be used to define an orthogonal coordinate system in the xy-plane. In Figure 1.1.4 the families $x^2 + y^2 = c^2$ and $y = kx$ are the coordinate curves of a polar coordinate system (that is, the curves $r =$ constant and $\theta =$ constant, respectively). In physics, the lines of electric force of a static configuration are the orthogonal trajectories of the family of equipotential curves. As a final example, if we consider a two-dimensional heated plate, then the heat energy flows along the orthogonal trajectories to the constant-temperature curves (isotherms).

Statement of the Problem: Given the equation of a family of curves, find the equation of the family of orthogonal trajectories.

Mathematical Formulation: We recall that curves that intersect at right angles satisfy the following:

The product of the slopes[3] at the point of intersection is -1.

Thus if the given family $F(x, y, c) = 0$ has slope $m_1 = f(x, y)$ at the point (x, y), then the slope of the family of orthogonal trajectories $G(x, y, k) = 0$ is $m_2 = -1/f(x, y)$, and therefore the differential equation that determines the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}.$$

[2]That is, the tangent lines to each curve are perpendicular at any point of intersection.
[3]By the slope of a curve at a given point, we mean the slope of the tangent line to the curve at that point.

Example 1.1.1 Determine the equation of the family of orthogonal trajectories to the curves with equation

$$y^2 = cx. \tag{1.1.12}$$

Solution: According to the preceding discussion, the differential equation determining the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{1}{f(x, y)},$$

where $f(x, y)$ denotes the slope of the given family at the point (x, y). To determine $f(x, y)$, we differentiate Equation (1.1.12) implicitly with respect to x to obtain

$$2y\frac{dy}{dx} = c. \tag{1.1.13}$$

We must now eliminate c from the previous equation to obtain an expression that gives the slope at the point (x, y). From Equation (1.1.12) we have

$$c = \frac{y^2}{x},$$

which, when substituted into Equation (1.1.13), yields

$$\frac{dy}{dx} = \frac{y}{2x}.$$

Consequently, the slope of the given family at the point (x, y) is

$$f(x, y) = \frac{y}{2x},$$

so that the orthogonal trajectories are obtained by solving the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}.$$

A key point to notice is that we cannot solve this differential equation by simply integrating with respect to x, since the function on the right-hand side of the differential equation depends on both x and y. However, multiplying by y, we see that

$$y\frac{dy}{dx} = -2x,$$

or equivalently,

$$\frac{d}{dx}\left(\frac{1}{2}y^2\right) = -2x.$$

Since the right-hand side of this equation depends only on x, whereas the term on the left-hand side is a derivative with respect to x, we can integrate both sides of the equation with respect to x to obtain

$$\frac{1}{2}y^2 = -x^2 + c_1,$$

which we write as

$$2x^2 + y^2 = k, \tag{1.1.14}$$

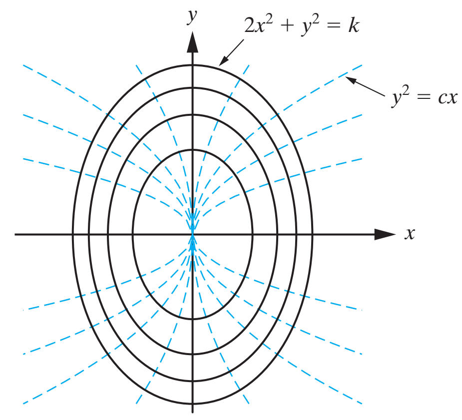

Figure 1.1.5: The family of curves $y^2 = cx$ and its orthogonal trajectories $2x^2 + y^2 = k$.

where $k = 2c_1$. We see that the curves in the given family (1.1.12) are parabolas, and the orthogonal trajectories (1.1.14) are a family of ellipses. This is illustrated in Figure 1.1.5.

□

Exercises for 1.1

Key Terms

Differential equation, Initial conditions, Initial-value problem, Newton's second law of motion, Hooke's law, Spring constant, Simple harmonic motion, Simple harmonic oscillator, Newton's law of cooling, Orthogonal trajectories.

Skills

- Given a differential equation, be able to check whether or not a given function $y = f(x)$ is indeed a solution to the differential equation.
- Be able to find the distance, velocity, and acceleration functions for an object moving freely under the influence of gravity.
- Be able to determine the motion of an object in a spring–mass system with no frictional or external forces.
- Be able to describe qualitatively how the temperature of an object changes as a function of time according to Newton's law of cooling.
- Be able to find the equation of the orthogonal trajectories to a given family of curves. In simple geometric cases, be prepared to provide rough sketches of some representative orthogonal trajectories.

True-False Review

For Questions 1–11, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. A differential equation for a function $y = f(x)$ must contain the first derivative $y' = f'(x)$.

2. The numerical values $y(0)$ and $y'(0)$ accompanying a differential equation for a function $y = f(x)$ are called initial conditions of the differential equation.

3. The relationship between the velocity and the acceleration of an object falling under the influence of gravity can be expressed mathematically as a differential equation.

4. A sketch of the height of an object falling freely under the influence of gravity as a function of time takes the shape of a parabola.

5. Hooke's law states that the restoring force of a spring is directly proportional to the displacement of the spring from its equilibrium position and is directed in the direction of the displacement from the equilibrium position.

6. If room temperature is 70°F, then an object whose temperature is 100°F at a particular time cools faster at that time than an object whose temperature at that time is 90°F.

7. According to Newton's law of cooling, the temperature of an object eventually becomes the same as the temperature of the surrounding medium.

8. A hot cup of coffee that is put into a cold room cools more in the first hour than the second hour.

9. At a point of intersection of a curve and one of its orthogonal trajectories, the slopes of the two curves are reciprocals of one another.

10. The family of orthogonal trajectories for a family of parallel lines is another family of parallel lines.

11. The family of orthogonal trajectories for a family of circles that are centered at the origin is another family of circles centered at the origin.

Problems

1. An object is released from rest at a height of 100 meters above the ground. Neglecting frictional forces, the subsequent motion is governed by the initial-value problem

$$\frac{d^2y}{dt^2} = g, \qquad y(0) = 0, \qquad \frac{dy}{dt}(0) = 0,$$

where $y(t)$ denotes the displacement of the object from its initial position at time t. Solve this initial-value problem and use your solution to determine the time when the object hits the ground.

2. A five-foot-tall boy tosses a tennis ball straight up from the level of the top of his head. Neglecting frictional forces, the subsequent motion is governed by the differential equation

$$\frac{d^2y}{dt^2} = g.$$

If the object hits the ground 8 seconds after the boy releases it, find

(a) the time when the tennis ball reaches its maximum height.

(b) the maximum height of the tennis ball.

3. A pyrotechnic rocket is to be launched vertically upward from the ground. For optimal viewing, the rocket should reach a maximum height of 90 meters above the ground. Ignore frictional forces.

(a) How fast must the rocket be launched in order to achieve optimal viewing?

(b) Assuming the rocket is launched with the speed determined in part (a), how long after it is launched will it reach its maximum height?

4. Repeat Problem 3 under the assumption that the rocket is launched from a platform 5 meters above the ground.

5. An object thrown vertically upward with a speed of 2 m/s from a height of h meters takes 10 seconds to reach the ground. Set up and solve the initial-value problem that governs the motion of the object, and determine h.

6. An object released from a height h meters above the ground with a vertical velocity of v_0 m/s hits the ground after t_0 seconds. Neglecting frictional forces, set up and solve the initial-value problem governing the motion, and use your solution to show that

$$v_0 = \frac{1}{2t_0}(2h - gt_0^2).$$

7. Verify that $y(t) = A\cos(\omega t - \phi)$ is a solution to the differential equation (1.1.7), where A, ω, and ϕ are constants with A and ω nonzero. Determine the constants A and ϕ (with $|\phi| < \pi$ radians) in the particular case when the initial conditions are

$$y(0) = a, \qquad \frac{dy}{dt}(0) = 0.$$

8. Verify that

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

is a solution to the differential equation (1.1.7). Show that the amplitude of the motion is

$$A = \sqrt{c_1^2 + c_2^2}.$$

9. Verify that, for $t > 0$, $y(t) = \ln t$ is a solution to the differential equation

$$2\left(\frac{dy}{dt}\right)^3 = \frac{d^3y}{dt^3}.$$

10. Verify that $y(x) = x/(x+1)$ is a solution to the differential equation

$$y + \frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{x^3 + 2x^2 - 3}{(1+x)^3}.$$

11. Verify that $y(x) = e^x \sin x$ is a solution to the differential equation

$$2y \cot x - \frac{d^2y}{dx^2} = 0.$$

12. By writing Equation (1.1.8) in the form

$$\frac{1}{T - T_m}\frac{dT}{dt} = -k$$

and using $u^{-1}\frac{du}{dt} = \frac{d}{dt}(\ln u)$, derive (1.1.9).

13. A glass of water whose temperature is 50°F is taken outside at noon on a day whose temperature is constant at 70°F. If the water's temperature is 55°F at 2 p.m., do you expect the water's temperature to reach 60°F before 4 p.m. or after 4 p.m.? Use Newton's law of cooling to explain your answer.

14. On a cold winter day (10°F), an object is brought outside from a 70°F room. If it takes 40 minutes for the object to cool from 70°F to 30°F, did it take more or less than 20 minutes for the object to reach 50°F? Use Newton's law of cooling to explain your answer.

For Problems 15–20, find the equation of the orthogonal trajectories to the given family of curves. In each case, sketch some curves from each family.

15. $x^2 + 4y^2 = c$.

16. $y = c/x$.

17. $y = cx^2$.

18. $y = cx^4$.

19. $y^2 = 2x + c$.

20. $y = ce^x$.

For Problems 21–24, m denotes a fixed nonzero constant, and c is the constant distinguishing the different curves in the given family. In each case, find the equation of the orthogonal trajectories.

21. $y = mx + c$.

22. $y = cx^m$.

23. $y^2 + mx^2 = c$.

24. $y^2 = mx + c$.

25. We call a coordinate system (u, v) **orthogonal** if its coordinate curves (the two families of curves u = constant and v = constant) are orthogonal trajectories (for example, a Cartesian coordinate system or a polar coordinate system). Let (u, v) be orthogonal coordinates, where $u = x^2 + 2y^2$, and x and y are Cartesian coordinates. Find the Cartesian equation of the v-coordinate curves, and sketch the (u, v) coordinate system.

26. Any curve with the property that whenever it intersects a curve of a given family it does so at an angle $a \neq \pi/2$ is called an **oblique trajectory** of the given family. (See Figure 1.1.6.) Let m_1 (equal to $\tan a_1$) denote the slope of the required family at the point (x, y), and let m_2 (equal to $\tan a_2$) denote the slope of the given family. Show that

$$m_1 = \frac{m_2 - \tan a}{1 + m_2 \tan a}.$$

[**Hint**: From Figure 1.1.6, $\tan a_1 = \tan(a_2 - a)$. Thus, the equation of the family of oblique trajectories is obtained by solving

$$\frac{dy}{dx} = \frac{m_2 - \tan a}{1 + m_2 \tan a}.]$$

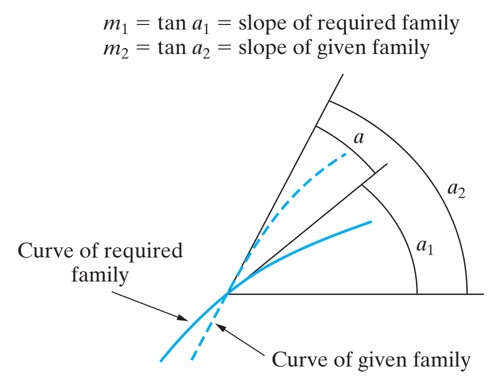

Figure 1.1.6: Oblique trajectories intersecting at an angle a.