3. Verify that for all values of t,

$$(1-t, 2+3t, 3-2t)$$

is a solution to the linear system

$$\begin{aligned} x_1 + x_2 + x_3 &= 6, \\ x_1 - x_2 - 2x_3 &= -7, \\ 5x_1 + x_2 - x_3 &= 4. \end{aligned}$$

4. Verify that for all values of s and t,

$$(s, s-2t, 2s+3t, t)$$

is a solution to the linear system

$$\begin{aligned} x_1 + x_2 - x_3 + 5x_4 &= 0, \\ 2x_2 - x_3 + 7x_4 &= 0, \\ 4x_1 + 2x_2 - 3x_3 + 13x_4 &= 0. \end{aligned}$$

5. By making a sketch in the xy-plane, prove that the following linear system has no solution:

$$\begin{aligned} 2x + 3y &= 1, \\ 2x + 3y &= 2. \end{aligned}$$

For Problems 6–8, determine the coefficient matrix, A, the right-hand-side vector, $\mathbf{b}$, and the augmented matrix $A^{\#}$ of the given system.

6. $$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 1, \\ 2x_1 + 4x_2 - 5x_3 &= 2, \\ 7x_1 + 2x_2 - x_3 &= 3. \end{aligned}$$

7. $$\begin{aligned} x + y + z - w &= 3, \\ 2x + 4y - 3z + 7w &= 2. \end{aligned}$$

8. $$\begin{aligned} x_1 + 2x_2 - x_3 &= 0, \\ 2x_1 + 3x_2 - 2x_3 &= 0, \\ 5x_1 + 6x_2 - 5x_3 &= 0. \end{aligned}$$

For Problems 9–10, write the system of equations with the given coefficient matrix and right-hand-side vector.

9. $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 1 & -2 & 6 \\ 3 & 1 & 4 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$

10. $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 2 \\ 7 & 6 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}.$

11. Consider the $m \times n$ homogeneous system of linear equations

$$A\mathbf{x} = \mathbf{0}. \qquad (2.3.2)$$

(a) If $\mathbf{x} = [x_1\ x_2\ \dots\ x_n]^T$ and $\mathbf{y} = [y_1\ y_2\ \dots\ y_n]^T$ are solutions to (2.3.2), show that

$$\mathbf{z} = \mathbf{x} + \mathbf{y} \quad \text{and} \quad \mathbf{w} = c\mathbf{x}$$

are also solutions, where c is an arbitrary scalar.

(b) Is the result of (a) true when $\mathbf{x}$ and $\mathbf{y}$ are solutions to the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$? Explain.

For Problems 12–15, write the vector formulation for the given system of differential equations.

12. $x_1' = -4x_1 + 3x_2 + 4t, \quad x_2' = 6x_1 - 4x_2 + t^2.$

13. $x_1' = t^2x_1 - tx_2, \quad x_2' = (-\sin t)x_1 + x_2.$

14. $x_1' = e^{2t}x_2, \quad x_2' + (\sin t)x_1 = 1.$

15. $x_1' = (-\sin t)x_2 + x_3 + t,\ x_2' = -e^t x_1 + t^2x_3 + t^3,$
$x_3' = -tx_1 + t^2x_2 + 1.$

For Problems 16–17 verify that the given vector function $\mathbf{x}$ defines a solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ for the given A and $\mathbf{b}$.

16. $\mathbf{x}(t) = \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{b}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

17. $\mathbf{x}(t) = \begin{bmatrix} 4e^{-2t} + 2\sin t \\ 3e^{-2t} - \cos t \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -4 \\ -3 & 2 \end{bmatrix},$
$\mathbf{b}(t) = \begin{bmatrix} -2(\cos t + \sin t) \\ 7\sin t + 2\cos t \end{bmatrix}.$

2.4 Elementary Row Operations and Row-Echelon Matrices

In the next section we will develop methods for solving a system of linear equations. These methods will consist of reducing a given system of equations to a new system that has the same solution set as the given system but is easier to solve. In this section we introduce the requisite mathematical results.

Elementary Row Operations

The first step in deriving systematic procedures for solving a linear system is to determine what operations can be performed on such a system without altering its solution set.

Example 2.4.1 Consider the system of equations

$$x_1 + 2x_2 + 4x_3 = 2, \tag{2.4.1}$$
$$2x_1 - 5x_2 + 3x_3 = 6, \tag{2.4.2}$$
$$4x_1 + 6x_2 - 7x_3 = 8. \tag{2.4.3}$$

Solution: If we permute (i.e., interchange), say, Equations (2.4.1) and (2.4.2), the resulting system is

$$\begin{aligned} 2x_1 - 5x_2 + 3x_3 &= 6,\\ x_1 + 2x_2 + 4x_3 &= 2,\\ 4x_1 + 6x_2 - 7x_3 &= 8, \end{aligned}$$

which certainly has the same solution set as the original system. Returning to the original system, if we multiply, say, Equation (2.4.2) by 5, we obtain the system

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 2,\\ 10x_1 - 25x_2 + 15x_3 &= 30,\\ 4x_1 + 6x_2 - 7x_3 &= 8, \end{aligned}$$

which again has the same solution set as the original system. Finally, if we add, say, twice Equation (2.4.1) to Equation (2.4.3), we obtain the system

$$x_1 + 2x_2 + 4x_3 = 2, \tag{2.4.4}$$
$$2x_1 - 5x_2 + 3x_3 = 6, \tag{2.4.5}$$
$$(4x_1 + 6x_2 - 7x_3) + 2(x_1 + 2x_2 + 4x_3) = 8 + 2(2). \tag{2.4.6}$$

We can verify that, if (2.4.4)–(2.4.6) are satisfied, then so are (2.4.1)–(2.4.3), and vice versa. It follows that the system of equations (2.4.4)–(2.4.6) has the same solution set as the original system of equations (2.4.1)–(2.4.3). □

More generally, similar reasoning can be used to show that the following three operations can be performed on any $m \times n$ system of linear equations without altering the solution set:

1. Permute equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of one equation to another equation.

Since these operations involve changes only in the system coefficients and constants (and not changes in the variables), they can be represented by the following operations on the rows of the augmented matrix of the system:

1. Permute rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of one row to another row.

These three operations, called **elementary row operations**, will be a basic computational tool throughout the text, even in cases when the matrix under consideration is not derived from a system of linear equations. The following notation will be used to describe elementary row operations performed on a matrix A.

1. P_{ij}: **P**ermute the ith and jth rows in A.
2. $\mathrm{M}_i(k)$: **M**ultiply every element of the ith row of A by a nonzero scalar k.
3. $\mathrm{A}_{ij}(k)$: **A**dd to the elements of the jth row of A the scalar k times the corresponding elements of the ith row of A.

Furthermore, the notation $A \sim B$ will mean that matrix B has been obtained from matrix A by a sequence of elementary row operations. To reference a particular elementary row operation used in, say, the nth step of the sequence of elementary row operations, we will write $\overset{n}{\sim} B$.

Example 2.4.2 The one-step operations performed on the system in Example 2.4.1 can be described as follows using elementary row operations on the augmented matrix of the system:

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & -5 & 3 & 6 \\ 4 & 6 & -7 & 8 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & -5 & 3 & 6 \\ 1 & 2 & 4 & 2 \\ 4 & 6 & -7 & 8 \end{bmatrix} \qquad 1.\ \mathrm{P}_{12}. \quad \text{Permute (2.4.1) and (2.4.2).}$$

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & -5 & 3 & 6 \\ 4 & 6 & -7 & 8 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 4 & 2 \\ 10 & -25 & 15 & 30 \\ 4 & 6 & -7 & 8 \end{bmatrix} \qquad 1.\ \mathrm{M}_{2}(5). \quad \text{Multiply (2.4.2) by 5.}$$

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & -5 & 3 & 6 \\ 4 & 6 & -7 & 8 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & -5 & 3 & 6 \\ 6 & 10 & 1 & 12 \end{bmatrix} \qquad 1.\ \mathrm{A}_{13}(2). \quad \text{Add 2 times (2.4.1) to (2.4.3).}$$

□

It is important to realize that each elementary row operation is reversible; we can "undo" a given elementary row operation by another elementary row operation to bring the modified linear system back into its original form. Specifically, in terms of the notation introduced above, the reverse operations are determined as follows (ERO refers here to "elementary row operation"):

ERO Applied to A $A \sim B$	**Reverse ERO Applied to B** $B \sim A$
P_{ij}	P_{ji}: Permute row j and i in B.
$\mathrm{M}_i(k)$	$\mathrm{M}_i(1/k)$: Multiply the ith row of B by $1/k$.
$\mathrm{A}_{ij}(k)$	$\mathrm{A}_{ij}(-k)$: Add to the elements of the jth row of B the scalar $-k$ times the corresponding elements of the ith row of B

We introduce a special term for matrices that are related via elementary row operations.

DEFINITION 2.4.3

Let A be an $m \times n$ matrix. Any matrix obtained from A by a finite sequence of elementary row operations is said to be **row-equivalent** to A.

Thus, all of the matrices in the previous example are row-equivalent. Since elementary row operations do not alter the solution set of a linear system, we have the next theorem.

Theorem 2.4.4 Systems of linear equations with row-equivalent augmented matrices have the same solution sets.

Row-Echelon Matrices

Our methods for solving a system of linear equations will consist of using elementary row operations to reduce the augmented matrix of the given system to a simple form. But how simple a form should we aim for? In order to answer this question, consider the system

$$x_1 + x_2 - x_3 = 4, \tag{2.4.7}$$

$$x_2 - 3x_3 = 5, \tag{2.4.8}$$

$$x_3 = 2. \tag{2.4.9}$$

This system can be solved most easily as follows. From Equation (2.4.9), $x_3 = 2$. Substituting this value into Equation (2.4.8) and solving for x_2 yields $x_2 = 5 + 6 = 11$. Finally, substituting for x_3 and x_2 into Equation (2.4.7) and solving for x_1, we obtain $x_1 = -5$. Thus, the solution to the given system of equations is $(-5, 11, 2)$, a single vector in $\mathbb{R}^3$. This technique is called **back substitution** and could be used because the given system has a simple form. The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

We see that the submatrix consisting of the first three columns (which corresponds to the matrix of coefficients) is an upper triangular matrix with the leftmost nonzero entry in each row equal to 1. The back-substitution method will work on any system of linear equations with an augmented matrix of this form. Unfortunately, not all systems of equations have augmented matrices that can be reduced to such a form. However, there is a simple type of matrix to which any matrix can be reduced by elementary row operations, and which also represents a system of equations that can be solved (if it has a solution) by back substitution. This is called a *row-echelon matrix* and is defined as follows:

DEFINITION 2.4.5

An $m \times n$ matrix is called a **row-echelon matrix** if it satisfies the following three conditions:

1. If there are any rows consisting entirely of zeros, they are grouped together at the bottom of the matrix.
2. The first nonzero element in any nonzero row[4] is a 1 (called a **leading 1**).
3. The leading 1 of any row below the first row is to the right of the leading 1 of the row above it.

[4] A *nonzero row* (*nonzero column*) is any row (column) that does not consist entirely of zeros.

Example 2.4.6 Examples of row-echelon matrices are

$$\begin{bmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & -1 & 6 & 5 & 9 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

whereas

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

are not row-echelon matrices. □

The basic result that will allow us to determine the solution set to any system of linear equations is stated in the next theorem.

Theorem 2.4.7 Any matrix is row-equivalent to a row-echelon matrix.

According to this theorem, by applying an appropriate sequence of elementary row operations to any $m \times n$ matrix, we can always reduce it to a row-echelon matrix. When a matrix A has been reduced to a row-echelon matrix in this way, we say that it has been reduced to **row-echelon form** and refer to the resulting matrix as a row-echelon form of A. The proof of Theorem 2.4.7 consists of giving an algorithm that will reduce an arbitrary $m \times n$ matrix to a row-echelon matrix after a finite sequence of elementary row operations. Before presenting such an algorithm, we first illustrate the result with an example.

Example 2.4.8 Use elementary row operations to reduce $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$ to row-echelon form.

Solution: We show each step in detail.

Step 1: Put a leading 1 in the (1, 1) position.

This is most easily accomplished by permuting rows 1 and 2.

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

Step 2: Use the leading 1 to put zeros beneath it in column 1.

This is accomplished by adding appropriate multiples of row 1 to the remaining rows.

$$\overset{2}{\sim} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 1 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix} \qquad \text{Step 2 row operations:} \quad \begin{cases} \text{Add } -2 \text{ times row 1 to row 2.} \\ \text{Add 4 times row 1 to row 3.} \\ \text{Add } -2 \text{ times row 1 to row 4.} \end{cases}$$

Step 3: Put a leading 1 in the (2, 2) position.

We could accomplish this by multiplying row 2 by 1/3. However, this would introduce fractions into the matrix and thereby complicate the remaining computations. In hand calculations, fewer algebraic errors result if we avoid the use of fractions. In this case, we can obtain a leading 1 without the use of fractions by adding −1 times row 3 to row 2.

$$\overset{3}{\sim}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix} \qquad \text{Step 3 row operation: Add } -1 \text{ times row 3 to row 2.}$$

Step 4: Use the leading 1 in the (2, 2) position to put zeros beneath it in column 2.

We now add appropriate multiples of row 2 to the rows *beneath* it. For row-echelon form, we need not be concerned about the row above it, however.

$$\overset{4}{\sim}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 13 & 13 \\ 0 & 0 & 9 & 9 \end{bmatrix} \qquad \text{Step 4 row operations: } \begin{cases} \text{Add } -2 \text{ times row 2 to row 3.} \\ \text{Add } -2 \text{ times row 2 to row 4.} \end{cases}$$

Step 5: Put a leading 1 in the (3, 3) position.

This can be accomplished by multiplying row 3 by 1/13.

$$\overset{5}{\sim}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 9 & 9 \end{bmatrix}$$

Step 6: Use the leading 1 in the (3, 3) position to put zeros beneath it in column 3.

The appropriate row operation is to add −9 times row 3 to row 4.

$$\overset{6}{\sim}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is a row-echelon matrix, hence the given matrix has been reduced to row-echelon form. The specific operations used at each step are given next, using the notation introduced previously in this section. In future examples, we will simply indicate briefly the elementary row operation used at each step. The following shows this description for the present example.

1. P_{12} **2.** $A_{12}(-2),\ A_{13}(4),\ A_{14}(-2)$ **3.** $A_{32}(-1)$
4. $A_{23}(-2),\ A_{24}(-2)$ **5.** $M_3(1/13)$ **6.** $A_{34}(-9)$

□

Remarks

1. Notice that in steps 2 and 4 of the preceding example we have performed multiple elementary row operations of the type $A_{ij}(k)$ in a single step. With this one exception, the reader is strongly advised not to combine multiple elementary row operations into a single step, particularly when they are of *different* types. This is a common source of calculation errors.

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2. The reader may have noticed that the particular steps taken in the preceding example are not uniquely determined. For instance, we could have achieved a leading 1 in the (1, 1) position in step 1 by multiplying the first row by $1/2$, rather than permuting the first two rows. Therefore, we may have multiple strategies for reducing a matrix to row-echelon form, and indeed, many possible row-echelon forms for a given matrix A. In this particular case, we chose not to multiply the first row by $1/2$ in order to avoid introducing fractions into the calculations.

The reader is urged to study the foregoing example very carefully, since it illustrates the general procedure for reducing an $m \times n$ matrix to row-echelon form using elementary row operations. This procedure will be used repeatedly throughout the text. The idea behind reduction to row-echelon form is to start at the upper left-hand corner of the matrix and proceed downward and to the right in the matrix. The following algorithm formalizes the steps that reduce any $m \times n$ matrix to row-echelon form using a finite number of elementary row operations and thereby provides a proof of Theorem 2.4.7. An illustration of this algorithm is given in Figure 2.4.1.

Algorithm for Reducing an $m \times n$ Matrix A to Row-Echelon Form

1. Start with an $m \times n$ matrix A. If $A = 0$, go to step 7.
2. Determine the leftmost nonzero column (this is called a **pivot column**, and the topmost position in this column is called a **pivot position**).
3. Use elementary row operations to put a 1 in the pivot position.
4. Use elementary row operations to put zeros below the pivot position.
5. If there are no more nonzero rows below the pivot position go to step 7, otherwise go to step 6.
6. Apply steps 2 through 5 to the submatrix consisting of the rows that lie below the pivot position.
7. The matrix is a row-echelon matrix.

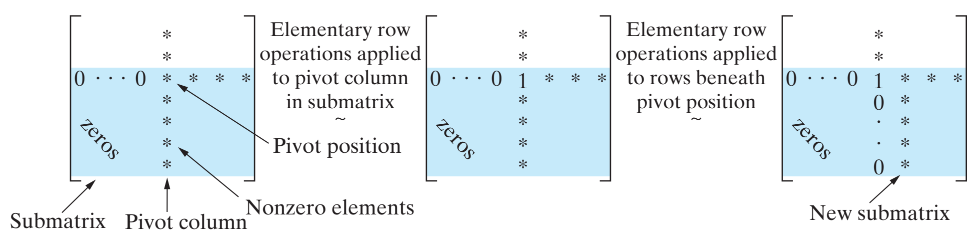

Figure 2.4.1: Illustration of an algorithm for reducing an $m \times n$ matrix to row-echelon form.

Remark In order to obtain a row-echelon matrix, we put a 1 in each pivot position. However, many algorithms for solving systems of linear equations numerically are based around the preceding algorithm, except that in step 3 we place a nonzero number (not necessarily a 1) in the pivot position. Of course, the matrix resulting from an application of this algorithm differs from a row-echelon matrix, since it will have arbitrary nonzero elements in the pivot positions.

Example 2.4.9 Reduce $\begin{bmatrix} 3 & 2 & -5 & 2 \\ 1 & 1 & -2 & 1 \\ 1 & 0 & -3 & 4 \end{bmatrix}$ to row-echelon form.

Solution: Applying the row-reduction algorithm leads to the following sequence of elementary row operations. The specific row operations used at each step are given at the end of the process.

$$\begin{bmatrix} \textcircled{3} & 2 & -5 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -3 & 4 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 2 & -5 & 2 \\ 1 & 0 & -3 & 4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & \textcircled{-1} & -2 & -1 \\ 0 & -1 & -2 & 3 \end{bmatrix}$$

(Pivot position; Pivot column; Pivot position; Pivot column)

$$\overset{3}{\sim} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & \textcircled{4} \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Pivot column; Pivot position)

This is a row-echelon matrix and hence we are done. The row operations used are summarized here:

1. P_{12} **2.** $A_{12}(-3)$, $A_{13}(-1)$ **3.** $M_2(-1)$ **4.** $A_{23}(1)$ **5.** $M_3(1/4)$ □

The Rank of a Matrix

We now derive some further results on row-echelon matrices that will be required in the next section to develop the theory for solving systems of linear equations.

We first observe that a row-echelon form for a matrix A is not unique. Given one row-echelon form for A, we can always obtain a different one by taking the first row-echelon form for A and adding some multiple of a given row to any rows above it. The result is still in row-echelon form.

However, even though the row-echelon form of A is not unique, we do have the following theorem (in Chapter 4 we will see how the proof of this theorem arises naturally from the more sophisticated ideas from linear algebra yet to be introduced).

Theorem 2.4.10 Let A be an $m \times n$ matrix. All row-echelon matrices that are row-equivalent to A have the same number of nonzero rows.

Theorem 2.4.10 associates a number with any $m \times n$ matrix A—namely, the number of nonzero rows in any row-echelon form of A. As we will see in the next section, this number is fundamental in determining the solution properties of linear systems, and indeed it plays a central role in linear algebra in general. For this reason, we give it a special name.

DEFINITION 2.4.11

The number of nonzero rows in any row-echelon form of a matrix A is called the **rank** of A and is denoted rank(A).

Example 2.4.12 Determine rank(A) if $A = \begin{bmatrix} 3 & 1 & 4 \\ 4 & 3 & 5 \\ 2 & -1 & 3 \end{bmatrix}$.

Solution: In order to determine rank(A), we must first reduce A to row-echelon form.

$$\begin{bmatrix} 3 & 1 & 4 \\ 4 & 3 & 5 \\ 2 & -1 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 4 & 3 & 5 \\ 2 & -1 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 1 \\ 0 & -5 & 1 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -\frac{1}{5} \\ 0 & -5 & 1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there are two nonzero rows in this row-echelon form of A, it follows from Definition 2.4.11 that rank$(A) = 2$.

1. $A_{31}(-1)$ **2.** $A_{12}(-4)$, $A_{13}(-2)$ **3.** $M_2(-1/5)$ **4.** $A_{23}(5)$

□

In the preceding example, the original matrix A had three nonzero rows, whereas any row-echelon form of A has only two nonzero rows. We can interpret this geometrically as follows. The three row vectors of A can be considered as vectors in $\mathbb{R}^3$ with components

$$\mathbf{a}_1 = (3, 1, 4), \quad \mathbf{a}_2 = (4, 3, 5), \quad \mathbf{a}_3 = (2, -1, 3).$$

In performing elementary row operations on A, we are taking combinations of these vectors in the following way:

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3,$$

and thus the rows of a row-echelon form of A are all of this form. We have been combining the vectors *linearly*. The fact that we obtained a row of zeros in the row-echelon form means that there are values of the constants c_1, c_2, c_3 such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = \mathbf{0},$$

where $\mathbf{0}$ denotes the zero vector (0, 0, 0). Equivalently, one of the vectors can be written in terms of the other two vectors, and therefore the three vectors lie in a plane. Reducing the matrix to row-echelon form has uncovered this relationship among the three vectors. We shall have much more to say about this in Chapter 4.

Remark If A is an $m \times n$ matrix, then rank$(A) \leq m$ and rank$(A) \leq n$. This is because the number of nonzero rows in a row-echelon form of A is equal to the number of pivots in a row-echelon form of A, which cannot exceed the number of rows or columns of A, since there can be at most one pivot per row and per column.

Reduced Row-Echelon Matrices

In the future we will need to consider the special row-echelon matrices that arise when zeros are placed above, as well as beneath, each leading 1. Any such matrix is called a reduced row-echelon matrix and is defined precisely as follows.

DEFINITION 2.4.13

An $m \times n$ matrix is called a **reduced row-echelon matrix** if it satisfies the following conditions:

1. It is a row-echelon matrix.
2. Any *column* that contains a leading 1 has zeros everywhere else.

Example 2.4.14 The following are examples of reduced row-echelon matrices:

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

Although an $m \times n$ matrix A does not have a unique row-echelon form, in reducing A to a reduced row-echelon matrix we are making a particular choice of row-echelon matrix, since we arrange that all elements above each leading 1 are zeros. In view of this, the following theorem should not be too surprising:

Theorem 2.4.15 An $m \times n$ matrix is row-equivalent to a *unique* reduced row-echelon matrix.

The unique reduced row-echelon matrix to which a matrix A is row-equivalent will be called *the* reduced row-echelon form of A. As illustrated in the next example, the row-reduction algorithm is easily extended to determine the reduced row-echelon form of A—we just put zeros above and beneath each leading 1.

Example 2.4.16 Determine the reduced row-echelon form of $A = \begin{bmatrix} 3 & -1 & 22 \\ -1 & 5 & 2 \\ 2 & 4 & 24 \end{bmatrix}$.

Solution: We apply the row-reduction algorithm, but put 0s above and below the leading 1s. In so doing, it is immaterial whether we first reduce A to row-echelon form and then arrange 0s above the leading 1s, or arrange 0s both above and below the leading 1s as we proceed from left to right.

$$A = \begin{bmatrix} 3 & -1 & 22 \\ -1 & 5 & 2 \\ 2 & 4 & 24 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 9 & 26 \\ -1 & 5 & 2 \\ 2 & 4 & 24 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 9 & 26 \\ 0 & 14 & 28 \\ 0 & -14 & -28 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 9 & 26 \\ 0 & 1 & 2 \\ 0 & -14 & -28 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

which is the reduced row-echelon form of A.

1. $A_{21}(2)$ **2.** $A_{12}(1)$, $A_{13}(-2)$ **3.** $M_2(1/14)$ **4.** $A_{21}(-9)$, $A_{23}(14)$ $\square$

Exercises for 2.4

Key Terms

Elementary row operations, Row-equivalent matrices, Back substitution, Row-echelon matrix, Row-echelon form, Leading 1, Pivot, Rank of a matrix, Reduced row-echelon matrix.

Skills

- Be able to perform elementary row operations on a matrix.
- Be able to determine a row-echelon form or reduced row-echelon form for a matrix.
- Be able to find the rank of a matrix.

True-False Review

For Questions 1–9, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. A matrix A can have many row-echelon forms but only one reduced row-echelon form.

2. Any upper triangular $n \times n$ matrix is in row-echelon form.

3. Any $n \times n$ matrix in row-echelon form is upper triangular.

4. If a matrix A has more rows than a matrix B, then $\text{rank}(A) \geq \text{rank}(B)$.

5. For any matrices A and B of the same dimensions,

$$\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B).$$

6. For any matrices A and B of the appropriate dimensions,

$$\text{rank}(AB) = \text{rank}(A) \cdot \text{rank}(B).$$

7. If a matrix has rank zero, then it must be the zero matrix.

8. The matrices A and $2A$ must have the same rank.

9. The matrices A and $2A$ must have the same reduced row-echelon form.

Problems

For Problems 1–8, determine whether the given matrices are in reduced row-echelon form, row-echelon form but not reduced row-echelon form, or neither.

1. $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

3. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

4. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

5. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

7. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

8. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

For Problems 9–18, use elementary row operations to reduce the given matrix to row-echelon form, and hence determine the rank of each matrix.

9. $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$.

10. $\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$.

11. $\begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix}$.

12. $\begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 3 & 5 \end{bmatrix}$.

13. $\begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$.

14. $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

15. $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 1 & -2 & 1 & 3 \\ 1 & -5 & 0 & 5 \end{bmatrix}$.

16. $\begin{bmatrix} 2 & -2 & -1 & 3 \\ 3 & -2 & 3 & 1 \\ 1 & -1 & 1 & 0 \\ 2 & -1 & 2 & 2 \end{bmatrix}$.

17. $\begin{bmatrix} 4 & 7 & 4 & 7 \\ 3 & 5 & 3 & 5 \\ 2 & -2 & 2 & -2 \\ 5 & -2 & 5 & -2 \end{bmatrix}$.

18. $\begin{bmatrix} 2 & 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 & 3 \\ 2 & 3 & 1 & 5 & 7 \end{bmatrix}$.

For Problems 19–25, reduce the given matrix to reduced row-echelon form and hence determine the rank of each matrix.

19. $\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$.

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20. $\begin{bmatrix} 3 & 7 & 10 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

21. $\begin{bmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \\ 6 & -6 & 12 \end{bmatrix}$.

22. $\begin{bmatrix} 3 & 5 & -12 \\ 2 & 3 & -7 \\ -2 & -1 & 1 \end{bmatrix}$.

23. $\begin{bmatrix} 1 & -1 & -1 & 2 \\ 3 & -2 & 0 & 7 \\ 2 & -1 & 2 & 4 \\ 4 & -2 & 3 & 8 \end{bmatrix}$.

24. $\begin{bmatrix} 1 & -2 & 1 & 3 \\ 3 & -6 & 2 & 7 \\ 4 & -8 & 3 & 10 \end{bmatrix}$.

25. $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$.

Many forms of technology have commands for performing elementary row operations on a matrix A. For example, in the linear algebra package of Maple, the three elementary row operations are

- swaprow(A, i, j): permute rows i and j
- mulrow(A, i, k): multiply row i by k
- addrow(A, i, j): add k times row i to row j

◇ For Problems 26–28, use some form of technology to determine a row-echelon form of the given matrix.

26. The matrix in Problem 14.

27. The matrix in Problem 15.

28. The matrix in Problem 18.

◇ Many forms of technology also have built-in functions for directly determining the reduced row-echelon form of a given matrix A. For example, in the linear algebra package of Maple, the appropriate command is rref(A). In Problems 29–31, use technology to determine directly the reduced row-echelon form of the given matrix.

29. The matrix in Problem 21.

30. The matrix in Problem 24.

31. The matrix in Problem 25.

2.5 Gaussian Elimination

We now illustrate how elementary row-operations applied to the augmented matrix of a system of linear equations can be used first to determine whether the system is consistent, and second, if the system is consistent, to find all of its solutions. In doing so, we will develop the general theory for linear systems of equations.

Example 2.5.1 Determine the solution set to

$$\begin{aligned} 3x_1 - 2x_2 + 2x_3 &= 9, \\ x_1 - 2x_2 + x_3 &= 5, \\ 2x_1 - x_2 - 2x_3 &= -1. \end{aligned} \tag{2.5.1}$$

Solution: We first use elementary row operations to reduce the augmented matrix of the system to row-echelon form.

$$\begin{bmatrix} 3 & -2 & 2 & 9 \\ 1 & -2 & 1 & 5 \\ 2 & -1 & -2 & -1 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -2 & 1 & 5 \\ 3 & -2 & 2 & 9 \\ 2 & -1 & -2 & -1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 4 & -1 & -6 \\ 0 & 3 & -4 & -11 \end{bmatrix}$$

$$\stackrel{3}{\sim} \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 3 & -4 & -11 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -13 & -26 \end{bmatrix} \stackrel{5}{\sim} \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$