CHAPTER 3

Determinants

Mathematics is the gate and key to the sciences. — Roger Bacon

In this chapter, we introduce a basic tool in applied mathematics, namely the determinant of a square matrix. The determinant is a number, associated with an $n \times n$ matrix A, whose value characterizes when the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution (or, equivalently, when A^{-1} exists). Determinants enjoy a wide range of applications, including coordinate geometry and function theory.

Sections 3.1–3.3 give a detailed introduction to determinants, their properties, and their applications. Alternatively, Section 3.4, "Summary of Determinants," can provide a nonrigorous and much more abbreviated introduction to the fundamental results required in the remainder of the text. We will see in later chapters that determinants are invaluable in the theory of eigenvalues and eigenvectors of a matrix, as well as in solution techniques for linear systems of differential equations.

3.1 The Definition of the Determinant

We will give a criterion shortly (Theorem 3.2.4) for the invertibility of a square matrix A in terms of the determinant of A, written $\det(A)$, which is a number determined directly from the elements of A. This criterion will provide a first extension of the Invertible Matrix Theorem introduced in Section 2.8.

To motivate the definition of the determinant of an $n \times n$ matrix A, we begin with the special cases $n = 1$, $n = 2$, and $n = 3$.

Case 1: $n = 1$. According to Theorem 2.6.5, the 1×1 matrix $A = [a_{11}]$ is invertible if and only if $\text{rank}(A) = 1$, if and only if the 1×1 determinant, $\det(A)$, defined by

$$\boxed{\det(A) = a_{11}}$$

is nonzero.

Case 2: $n = 2$. According to Theorem 2.6.5, the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is invertible if and only if $\text{rank}(A) = 2$, if and only if the row-echelon form of A has two nonzero rows. Provided that $a_{11} \neq 0$, we can reduce A to row-echelon form as follows:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \dfrac{a_{12}a_{21}}{a_{11}} \end{bmatrix}.$$

$$\boxed{\mathbf{1}.\ A_{12}\left(-\frac{a_{21}}{a_{11}}\right)}$$

For A to be invertible, it is necessary that $a_{22} - \dfrac{a_{12}a_{21}}{a_{11}} \neq 0$, or that $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Thus, for A to be invertible, it is necessary that the 2×2 determinant, $\det(A)$, defined by

$$\boxed{\det(A) = a_{11}a_{22} - a_{12}a_{21}} \tag{3.1.1}$$

be nonzero. We will see in the next section that this condition is also sufficient for the 2×2 matrix A to be invertible.

Case 3: $n = 3$. According to Theorem 2.6.5, the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is invertible if and only if $\text{rank}(A) = 3$, if and only if the row-echelon form of A has three nonzero rows. Reducing A to row-echelon form as in Case 2, we find that it is necessary that the 3×3 determinant defined by

$$\boxed{\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}} \tag{3.1.2}$$

be nonzero. Again, in the next section we will prove that this condition on $\det(A)$ is also sufficient for the 3×3 matrix A to be invertible.

To generalize the foregoing formulas for the determinant of an $n \times n$ matrix A, we take a closer look at their structure. Each determinant above consists of a sum of $n!$ products, where each product term contains precisely one element from each row and each column of A. Furthermore, each possible choice of one element from each row and each column of A does in fact occur as a term of the summation. Finally, each term is assigned a plus or a minus sign. Based on these observations, the appropriate way in which to define $\det(A)$ for an $n \times n$ matrix would seem to be to add up all possible products consisting of one element from each row and each column of A, with some condition on which products are taken with a plus sign and which with a minus sign. To describe this condition, we digress to discuss permutations.

Permutations

Consider the first n positive integers $1, 2, 3, \ldots, n$. Any arrangement of these integers in a specific order, say, $(p_1, p_2, \ldots, p_n)$, is called a **permutation**.

Example 3.1.1 There are precisely six distinct permutations of the integers 1, 2 and 3:

$$(1, 2, 3),\quad (1, 3, 2),\quad (2, 1, 3),\quad (2, 3, 1),\quad (3, 1, 2),\quad (3, 2, 1). \qquad \square$$

More generally, we have the following result:

Theorem 3.1.2 There are precisely $n!$ distinct permutations of the integers $1, 2, \ldots, n$.

The proof of this result is left as an exercise.

The elements in the permutation $(1, 2, \ldots, n)$ are said to be in their natural increasing order. We now introduce a number that describes how far a given permutation is from its natural order. For $i \neq j$, the pair of elements p_i and p_j in the permutation $(p_1, p_2, \ldots, p_n)$ are said to be **inverted** if they are out of their natural order; that is, if $p_i > p_j$ with $i < j$. If this is the case, we say that (p_i, p_j) is an **inversion**. For example, in the permutation $(4, 2, 3, 1)$, the pairs $(4, 2)$, $(4, 3)$, $(4, 1)$, $(2, 1)$, and $(3, 1)$ are all out of their natural order. Consequently, there are a total of five inversions in this permutation. In general we let $N(p_1, p_2, \ldots, p_n)$ denote the total number of inversions in the permutation $(p_1, p_2, \ldots, p_n)$.

Example 3.1.3 Find the number of inversions in the permutations $(1, 3, 2, 4, 5)$ and $(2, 4, 5, 3, 1)$.

Solution: The only pair of elements in the permutation $(1, 3, 2, 4, 5)$ that is out of natural order is $(3, 2)$, so $N(1, 3, 2, 4, 5) = 1$.

The permutation $(2, 4, 5, 3, 1)$ has the following pairs of elements out of natural order: $(2, 1)$, $(4, 3)$, $(4, 1)$, $(5, 3)$, $(5, 1)$, and $(3, 1)$. Thus, $N(2, 4, 5, 3, 1) = 6$. $\square$

It can be shown that the number of inversions gives the minimum number of adjacent interchanges of elements in the permutation that are required to restore the permutation to its natural increasing order. This justifies the claim that the number of inversions describes how far from natural order a given permutation is. For example, $N(3, 2, 1) = 3$, and the permutation $(3, 2, 1)$ can be restored to its natural order by the following sequence of adjacent interchanges:

$$(3, 2, 1) \to (3, 1, 2) \to (1, 3, 2) \to (1, 2, 3).$$

The number of inversions enables us to distinguish two different types of permutations as follows.

DEFINITION 3.1.4

1. If $N(p_1, p_2, \ldots, p_n)$ is an even integer (or zero), we say $(p_1, p_2, \ldots, p_n)$ is an **even permutation**. We also say that $(p_1, p_2, \ldots, p_n)$ has **even parity**.
2. If $N(p_1, p_2, \ldots, p_n)$ is an odd integer, we say $(p_1, p_2, \ldots, p_n)$ is an **odd permutation**. We also say that $(p_1, p_2, \ldots, p_n)$ has **odd parity**.

Example 3.1.5 The permutation (4, 1, 3, 2) has even parity, since we have $N(4, 1, 3, 2) = 4$, whereas (3, 2, 1, 4) is an odd permutation since $N(3, 2, 1, 4) = 3$. □

We associate a plus or a minus sign with a permutation, depending on whether it has even or odd parity, respectively. The sign associated with the permutation $(p_1, p_2, \ldots, p_n)$ can be specified by the indicator $\sigma(p_1, p_2, \ldots, p_n)$, defined in terms of the number of inversions as follows:

$$\sigma(p_1, p_2, \ldots, p_n) = \begin{cases} +1 & \text{if } (p_1, p_2, \ldots, p_n) \text{ has even parity,} \\ -1 & \text{if } (p_1, p_2, \ldots, p_n) \text{ has odd parity.} \end{cases}$$

Hence,

$$\boxed{\sigma(p_1, p_2, \ldots, p_n) = (-1)^{N(p_1, p_2, \ldots, p_n)}.}$$

Example 3.1.6 It follows from Example 3.1.3 that

$$\sigma(1, 3, 2, 4, 5) = (-1)^1 = -1,$$

whereas

$$\sigma(2, 4, 5, 3, 1) = (-1)^6 = 1.$$ □

The proofs of some of our later results will depend upon the next theorem.

Theorem 3.1.7 If any two elements in a permutation are interchanged, then the parity of the resulting permutation is opposite to that of the original permutation.

Proof We first show that interchanging two adjacent terms in a permutation changes its parity. Consider an arbitrary permutation $(p_1, \ldots, p_k, p_{k+1}, \ldots, p_n)$, and suppose we interchange the adjacent elements p_k and p_{k+1}. Then

- If $p_k > p_{k+1}$, then

$$N(p_1, p_2, \ldots, p_{k+1}, p_k, \ldots, p_n) = N(p_1, p_2, \ldots, p_k, p_{k+1}, \ldots, p_n) - 1,$$

- If $p_k < p_{k+1}$, then

$$N(p_1, p_2, \ldots, p_{k+1}, p_k, \ldots, p_n) = N(p_1, p_2, \ldots, p_k, p_{k+1}, \ldots, p_n) + 1,$$

so that the parity is changed in both cases.

Now suppose we interchange the elements p_i and p_k in the permutation $(p_1, p_2, \ldots, p_i, \ldots, p_k, \ldots, p_n)$. Note that $k - i > 0$. We can accomplish this by successively interchanging adjacent elements. In moving p_k to the ith position, we perform $k - i$ interchanges involving adjacent terms, and the resulting permutation is

$$(p_1, p_2, \ldots, p_k, p_i, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n).$$

Next we move p_i to the kth position. A moment's thought shows that this requires $(k-i)-1$ interchanges of adjacent terms. Thus, the total number of adjacent interchanges involved in interchanging the elements p_i and p_k is $2(k - i) - 1$, which is always an odd integer. Since each adjacent interchange changes the parity, the permutation resulting from an odd number of adjacent interchanges has opposite parity to the original permutation. ■

At this point, we are ready to see how permutations can facilitate the definition of the determinant. From the expression (3.1.2) for the 3×3 determinant, we see that the row indices of each term have been arranged in their natural increasing order and that the column indices are each a permutation (p_1, p_2, p_3) of 1, 2, 3. Further, the sign attached to each term coincides with the sign of the permutation of the corresponding column indices; that is, $\sigma(p_1, p_2, p_3)$. These observations motivate the following general definition of the determinant of an $n \times n$ matrix:

DEFINITION 3.1.8

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **determinant of** A, denoted $\det(A)$, is defined as follows:

$$\det(A) = \sum \sigma(p_1, p_2, \ldots, p_n) a_{1p_1} a_{2p_2} a_{3p_3} \cdots a_{np_n}, \tag{3.1.3}$$

where the summation is over the $n!$ distinct permutations $(p_1, p_2, \ldots, p_n)$ of the integers $1, 2, 3, \ldots, n$. The determinant of an $n \times n$ matrix is said to have **order** n.

We sometimes denote $\det(A)$ by

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}.$$

Thus, for example, from (3.1.1), we have

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example 3.1.9 Use Definition 3.1.8 to derive the expression for the determinant of order 3.

Solution: When $n = 3$, (3.1.3) reduces to

$$\det(A) = \sum \sigma(p_1, p_2, p_3) a_{1p_1} a_{2p_2} a_{3p_3},$$

where the summation is over the $3! = 6$ permutations of 1, 2, 3. It follows that the six terms in this summation are

$$a_{11}a_{22}a_{33}, \quad a_{11}a_{23}a_{32}, \quad a_{12}a_{21}a_{33}, \quad a_{12}a_{23}a_{31}, \quad a_{13}a_{21}a_{32}, \quad a_{13}a_{22}a_{31},$$

so that

$$\det(A) = \sigma(1, 2, 3)a_{11}a_{22}a_{33} + \sigma(1, 3, 2)a_{11}a_{23}a_{32} + \sigma(2, 1, 3)a_{12}a_{21}a_{33} + \sigma(2, 3, 1)a_{12}a_{23}a_{31} + \sigma(3, 1, 2)a_{13}a_{21}a_{32} + \sigma(3, 2, 1)a_{13}a_{22}a_{31}.$$

To obtain the values of each $\sigma(p_1, p_2, p_3)$, we determine the parity for each permutation (p_1, p_2, p_3). We find that

$$\sigma(1, 2, 3) = +1, \quad \sigma(1, 3, 2) = -1, \quad \sigma(2, 1, 3) = -1,$$
$$\sigma(2, 3, 1) = +1, \quad \sigma(3, 1, 2) = +1, \quad \sigma(3, 2, 1) = -1.$$

Hence,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

□

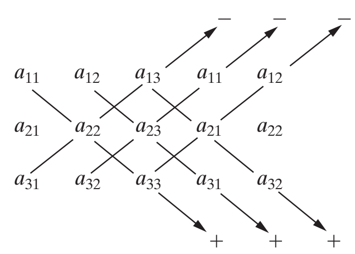

Figure 3.1.1: A schematic for obtaining the determinant of a 3×3 matrix $A = [a_{ij}]$.

A simple schematic for obtaining the terms in the determinant of order 3 is given in Figure 3.1.1. By taking the product of the elements joined by each arrow and attaching the indicated sign to the result, we obtain the six terms in the determinant of the 3×3 matrix $A = [a_{ij}]$. Note that this technique for obtaining the terms in a determinant *does not* generalize to determinants of $n \times n$ matrices with $n > 3$.

Example 3.1.10

Evaluate

$$\text{(a)}\quad |-3|. \qquad \text{(b)}\quad \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix}. \qquad \text{(c)}\quad \begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 2 \\ 0 & 3 & 1 \end{vmatrix}.$$

Solution:

(a) $|-3| = -3$. In the case of a 1×1 matrix, the reader is cautioned not to confuse the vertical bars notation for the determinant with absolute value bars.

(b) $\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = (3)(4) - (-2)(1) = 14.$

(c) In this case, the schematic in Figure 3.1.1 is

$$\begin{matrix} 1 & 2 & -3 & 1 & 2 \\ 4 & -1 & 2 & 4 & -1 \\ 0 & 3 & 1 & 0 & 3 \end{matrix}$$

so that

$$\begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = (1)(-1)(1) + (2)(2)(0)$$
$$+ (-3)(4)(3) - (0)(-1)(-3) - (3)(2)(1) - (1)(4)(2) = -51.$$

□

We now to turn to some geometric applications of the determinant.

Geometric Interpretation of the Determinants of Orders Two and Three

If $\mathbf{a}$ and $\mathbf{b}$ are two vectors in space, we recall that their dot product is the scalar

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}|| \cos\theta, \tag{3.1.4}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $||\mathbf{a}||$ and $||\mathbf{b}||$ denote the lengths of $\mathbf{a}$ and $\mathbf{b}$, respectively. On the other hand, the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta\ \mathbf{n}, \tag{3.1.5}$$

where $\mathbf{n}$ denotes a unit vector[1] that is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ and chosen in such a way that $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ is a right-handed set of vectors. If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the unit vectors pointing along the positive x-, y- and z-axes, respectively, of a rectangular Cartesian coordinate system and $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then Equation (3.1.5) can be expressed in component form as

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \tag{3.1.6}$$

This can be remembered most easily in the compact form

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

whose validity is readily checked by using the schematic in Figure 3.1.1. We will use the equations above to establish the following theorem.

Theorem 3.1.11

1. The area of a parallelogram with sides determined by the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ is
$$\text{Area} = |\det(A)|,$$
where $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$.
2. The volume of a parallelepiped determined by the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ is
$$\text{Volume} = |\det(A)|,$$
where $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$.

Before presenting the proof of this theorem, we make some remarks and give two examples.

Remarks

1. The vertical bars appearing in the formulas in Theorem 3.1.11 denote the absolute value of the number $\det(A)$.
2. We see from the expression for the volume of a parallelepiped that the condition for three vectors to lie in the same plane (i.e., the parallelepiped has zero volume) is that $\det(A) = 0$. This will be a useful result in the next chapter.

Example 3.1.12 Find the area of the parallelogram containing the points (0, 0), (1, 2), (3, 4) and (4, 6).

Solution: The sides of the parallelogram are determined by the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$. According to part 1 of Theorem 3.1.11, the area of the parallelogram is

$$\left|\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right| = |(1)(4) - (2)(3)| = |-2| = 2.$$ □

[1] A unit vector is a vector of length 1.

Example 3.1.13 Determine whether or not the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$, and $\mathbf{c} = -5\mathbf{i} + (-7)\mathbf{j} + (-9)\mathbf{k}$ lie in a single plane in 3-space.

Solution: By Remark 2 above, it suffices to determine whether or not the volume of the parallelepiped determined by the three vectors is zero or not. To do this, we use part 2 of Theorem 3.1.11:

$$\text{Volume} = \left|\det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -5 & -7 & -9 \end{bmatrix}\right|$$

$$= \left|\begin{matrix} (1)(5)(-9) + (2)(6)(-5) + (3)(4)(-7) \\ -(-5)(5)(3) - (-7)(6)(1) - (-9)(4)(2) \end{matrix}\right| = 0,$$

which shows that the three vectors do lie in a single plane. □

Now we turn to the

Proof of Theorem 3.1.11:

1. The area of the parallelogram is

$$\text{area} = (\text{length of base}) \times (\text{perpendicular height}).$$

From Figure 3.1.2, this can be written as

$$\text{Area} = ||\mathbf{a}||h = ||\mathbf{a}||\,||\mathbf{b}||\,|\sin\theta| = ||\mathbf{a} \times \mathbf{b}||. \quad (3.1.7)$$

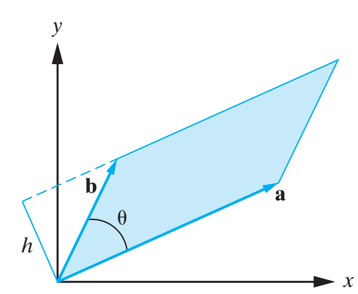

Figure 3.1.2: Determining the area of a parallelogram.

Since the $\mathbf{k}$ components of $\mathbf{a}$ and $\mathbf{b}$, a_3 and b_3, are both zero (since the vectors lie in the xy-plane), substitution from Equation (3.1.6) yields

$$\text{Area} = ||(a_1b_2 - a_2b_1)\mathbf{k}|| = |a_1b_2 - a_2b_1| = |\det(A)|.$$

2. The volume of the parallelepiped is

$$\text{Volume} = (\text{area of base}) \times (\text{perpendicular height}).$$

The base is determined by the vectors $\mathbf{b}$ and $\mathbf{c}$ (see Figure 3.1.3), and its area can be written as $||\mathbf{b} \times \mathbf{c}||$, in similar fashion to what was done in (3.1.7). From Figure 3.1.3 and Equation (3.1.4), we therefore have

$$\text{Volume} = ||\mathbf{b} \times \mathbf{c}||h = ||\mathbf{b} \times \mathbf{c}|| \, ||\mathbf{a}|| |\cos\psi| = ||\mathbf{b} \times \mathbf{c}|| \, |\mathbf{a} \cdot \mathbf{n}|,$$

where $\mathbf{n}$ is a unit vector that is perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{c}$. We can now use Equations (3.1.5) and (3.1.6) to obtain

$$\begin{aligned}\text{Volume} &= ||\mathbf{b} \times \mathbf{c}|| \, ||\mathbf{a}|| \, |\cos\psi| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \\ &= \left|(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot \left[(b_2c_3 - b_3c_2)\mathbf{i} + (b_3c_1 - b_1c_3)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}\right]\right| \\ &= |a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)| \\ &= |\det(A)|,\end{aligned}$$

as required.

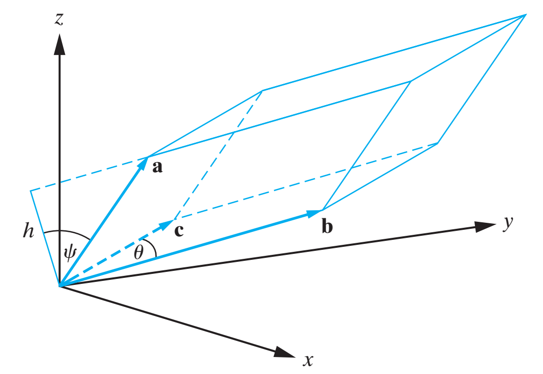

Figure 3.1.3: Determining the volume of a parallelepiped.

■

Exercises for 3.1

Key Terms

Permutation, Inversion, Parity, Determinant, Order, Dot product, Cross product.

Skills

- Be able to compute determinants by using Definition 3.1.8.
- Be able to list permutations of $1, 2, \dots, n$.
- Be able to find the number of inversions of a given permutation and thus determine its parity.
- Be able to compute the area of a parallelogram with sides determined by vectors in $\mathbb{R}^2$.
- Be able to compute the volume of a parallelogram with sides determined by vectors in $\mathbb{R}^3$.

True-False Review

For Questions 1–8, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. If A is a 2×2 lower triangular matrix, then $\det(A)$ is the product of the elements on the main diagonal of A.

2. If A is a 3×3 upper triangular matrix, then $\det(A)$ is the product of the elements on the main diagonal of A.

3. The volume of the parallelepiped whose sides are determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is given by $\det(A)$, where $A = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

4. There are the same number of permutations of $\{1, 2, 3, 4\}$ of even parity as there are of odd parity.

5. If A and B are 2×2 matrices, then $\det(A + B) = \det(A) + \det(B)$.

6. The determinant of a matrix whose elements are all positive must be positive.

7. A matrix containing a row of zeros must have zero determinant.

8. Three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^3$ are coplanar if and only if the determinant of the 3×3 matrix $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ is zero.

Problems

For Problems 1–6, determine the parity of the given permutation.

1. $(2, 1, 3, 4)$.

2. $(1, 3, 2, 4)$.

3. $(1, 4, 3, 5, 2)$.

4. $(5, 4, 3, 2, 1)$.

5. $(1, 5, 2, 4, 3)$.

6. $(2, 4, 6, 1, 3, 5)$.

7. Use the definition of a determinant to derive the general expression for the determinant of A if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

For Problems 8–15, evaluate the determinant of the given matrix.

8. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

9. $A = \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix}$.

10. $A = \begin{bmatrix} -4 & 10 \\ -1 & 8 \end{bmatrix}$.

11. $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 6 \\ 0 & 2 & -1 \end{bmatrix}$.

12. $A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 2 & 3 \\ 9 & 5 & 1 \end{bmatrix}$.

13. $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -4 & 1 \\ -1 & 5 & -7 \end{bmatrix}$.

14. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$.

15. $A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \end{bmatrix}$.

For Problems 16–21, evaluate the given determinant.

16. $\begin{vmatrix} \pi & \pi^2 \\ \sqrt{2} & 2\pi \end{vmatrix}$.

17. $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 4 & 1 \\ 3 & 1 & 6 \end{vmatrix}$.

18. $\begin{vmatrix} 3 & 2 & 6 \\ 2 & 1 & -1 \\ -1 & 1 & 4 \end{vmatrix}$.

19. $\begin{vmatrix} 2 & 3 & 6 \\ 0 & 1 & 2 \\ 1 & 5 & 0 \end{vmatrix}$.

20. $\begin{vmatrix} \sqrt{\pi} & e^2 & e^{-1} \\ \sqrt{67} & 1/30 & 2001 \\ \pi & \pi^2 & \pi^3 \end{vmatrix}$.

21. $\begin{vmatrix} e^{2t} & e^{3t} & e^{-4t} \\ 2e^{2t} & 3e^{3t} & -4e^{-4t} \\ 4e^{2t} & 9e^{3t} & 16e^{-4t} \end{vmatrix}$.

In Problems 22–23, we explore a relationship between determinants and solutions to a differential equation. The 3×3 matrix consisting of solutions to a differential equation and their derivatives is called the **Wronskian** and, as we will see in later chapters, plays a pivotal role in the theory of differential equations.

22. Verify that $y_1(x) = \cos 2x$, $y_2(x) = \sin 2x$, and $y_3(x) = e^x$ are solutions to the differential equation

$$y''' - y'' + 4y' - 4y = 0,$$

and show that $\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$ is nonzero on any interval.

23. **(a)** Verify that $y_1(x) = e^x$, $y_2(x) = \cosh x$, and $y_3(x) = \sinh x$ are solutions to the differential equation

$$y''' - y'' - y' + y = 0,$$

and show that $\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$ is identically zero.

(b) Determine nonzero constants d_1, d_2, and d_3 such that

$$d_1y_1 + d_2y_2 + d_3y_3 = 0.$$

24. **(a)** Write all 24 distinct permutations of the integers 1, 2, 3, 4.

(b) Determine the parity of each permutation in part (a).

(c) Use parts (a) and (b) to derive the expression for a determinant of order 4.

For Problems 25–27, use the previous problem to compute the determinant of A.

25. $A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 3 & 0 & 2 & 5 \\ 2 & 1 & 0 & 3 \\ 9 & -1 & 2 & 1 \end{bmatrix}$.

26. $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 3 & 1 & -2 & 3 \\ 2 & 3 & 1 & 2 \\ -2 & 3 & 5 & -2 \end{bmatrix}$.

27. $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 3 & 4 \\ 3 & 4 & 0 & 5 \\ 4 & 5 & 6 & 0 \end{bmatrix}$.

28. Use Problem 27 to find the determinant of A, where

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 \\ 2 & 0 & 3 & 4 & 0 \\ 3 & 4 & 0 & 5 & 0 \\ 4 & 5 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}.$$

29. **(a)** If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and c is a constant, verify that $\det(cA) = c^2 \det(A)$.

(b) Use the definition of a determinant to prove that if A is an $n \times n$ matrix and c is a constant, then $\det(cA) = c^n \det(A)$.

For Problems 30–33, determine whether the given expression is a term in the determinant of order 5. If it is, determine whether the permutation of the column indices has even or odd parity and hence find whether the term has a plus or a minus sign attached to it.

30. $a_{11}a_{25}a_{33}a_{42}a_{54}$.

31. $a_{11}a_{23}a_{34}a_{43}a_{52}$.

32. $a_{13}a_{25}a_{31}a_{44}a_{42}$.

33. $a_{11}a_{32}a_{24}a_{43}a_{55}$.

For Problems 34–37, determine the values of the indices p and q such that the following are terms in a determinant of order 4. In each case, determine the number of inversions in the permutation of the column indices and hence find the appropriate sign that should be attached to each term.

34. $a_{13}a_{p4}a_{32}a_{2q}$.

35. $a_{21}a_{3q}a_{p2}a_{43}$.

36. $a_{3q}a_{p4}a_{13}a_{42}$.

37. $a_{pq}a_{34}a_{13}a_{42}$.

38. The *alternating symbol* ϵ_{ijk} is defined by

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (ijk) \text{ is an even permutation of } 1, 2, 3, \\ -1, & \text{if } (ijk) \text{ is an odd permutation of } 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Write all nonzero ϵ_{ijk}, for $1 \le i \le 3$, $1 \le j \le 3$, $1 \le k \le 3$.

(b) If $A = [a_{ij}]$ is a 3×3 matrix, verify that

$$\det(A) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}a_{1i}a_{2j}a_{3k}.$$

39. If A is the general $n \times n$ matrix, determine the sign attached to the term

$$a_{1n}a_{2\,n-1}a_{3\,n-2}\cdots a_{n1},$$

which arises in $\det(A)$.

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40. ◇ Use some form of technology to evaluate the determinants in Problems 16–21.

41. ◇ Let A be an arbitrary 4×4 matrix. By experimenting with various elementary row operations, conjecture how elementary row operations applied to A affect the value of $\det(A)$.

42. ◇ Verify that $y_1(x) = e^{-2x} \cos 3x$, $y_2(x) = e^{-2x} \sin 3x$, and $y_3(x) = e^{-4x}$ are solutions to the differential equation

$$y''' + 8y'' + 29y' + 52y = 0,$$

and show that $\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$ is nonzero on any interval.

3.2 Properties of Determinants

For large values of n, evaluating a determinant of order n using the definition given in the previous section is not very practical, since the number of terms is $n!$ (for example, a determinant of order 10 contains 3,628,800 terms). In the next two sections, we develop better techniques for evaluating determinants. The following theorem suggests one way to proceed.

Theorem 3.2.1 If A is an $n \times n$ upper or lower triangular matrix, then

$$\det(A) = a_{11}a_{22}a_{33} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

Proof We use the definition of the determinant to prove the result in the upper triangular case. From Equation (3.1.3),

$$\det(A) = \sum \sigma(p_1, p_2, \dots, p_n) a_{1p_1} a_{2p_2} a_{3p_3} \dots a_{np_n}. \tag{3.2.1}$$

If A is upper triangular, then $a_{ij} = 0$ whenever $i > j$, and therefore the only nonzero terms in the preceding summation are those with $p_i \geq i$ for all i. Since all the p_i must be distinct, the only possibility is (by applying $p_i \geq i$ to $i = n, n-1, \dots, 2, 1$ in turn)

$$p_i = i, \qquad i = 1, 2, \dots, n,$$

and so Equation (3.2.1) reduces to the single term

$$\det(A) = \sigma(1, 2, \dots, n) a_{11} a_{22} \cdots a_{nn}.$$

Since $\sigma(1, 2, \dots, n) = 1$, it follows that

$$\det(A) = a_{11} a_{22} \cdots a_{nn}.$$

The proof in the lower triangular case is left as an exercise (Problem 47). ■

Example 3.2.2 According to the previous theorem,

$$\begin{vmatrix} 2 & 5 & -1 & 3 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 \end{vmatrix} = (2)(-1)(7)(5) = -70.$$ □