40. ◇ Use some form of technology to evaluate the determinants in Problems 16–21.

41. ◇ Let A be an arbitrary 4×4 matrix. By experimenting with various elementary row operations, conjecture how elementary row operations applied to A affect the value of $\det(A)$.

42. ◇ Verify that $y_1(x) = e^{-2x}\cos 3x$, $y_2(x) = e^{-2x}\sin 3x$, and $y_3(x) = e^{-4x}$ are solutions to the differential equation

$$y''' + 8y'' + 29y' + 52y = 0,$$

and show that $\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$ is nonzero on any interval.

3.2 Properties of Determinants

For large values of n, evaluating a determinant of order n using the definition given in the previous section is not very practical, since the number of terms is $n!$ (for example, a determinant of order 10 contains 3,628,800 terms). In the next two sections, we develop better techniques for evaluating determinants. The following theorem suggests one way to proceed.

Theorem 3.2.1 If A is an $n \times n$ upper or lower triangular matrix, then

$$\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

Proof We use the definition of the determinant to prove the result in the upper triangular case. From Equation (3.1.3),

$$\det(A) = \sum \sigma(p_1, p_2, \ldots, p_n) a_{1p_1}a_{2p_2}a_{3p_3}\ldots a_{np_n}. \tag{3.2.1}$$

If A is upper triangular, then $a_{ij} = 0$ whenever $i > j$, and therefore the only nonzero terms in the preceding summation are those with $p_i \geq i$ for all i. Since all the p_i must be distinct, the only possibility is (by applying $p_i \geq i$ to $i = n, n-1, \ldots, 2, 1$ in turn)

$$p_i = i, \qquad i = 1, 2, \ldots, n,$$

and so Equation (3.2.1) reduces to the single term

$$\det(A) = \sigma(1, 2, \ldots, n) a_{11}a_{22}\cdots a_{nn}.$$

Since $\sigma(1, 2, \ldots, n) = 1$, it follows that

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

The proof in the lower triangular case is left as an exercise (Problem 47). ■

Example 3.2.2 According to the previous theorem,

$$\begin{vmatrix} 2 & 5 & -1 & 3 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 \end{vmatrix} = (2)(-1)(7)(5) = -70.$$ □

Theorem 3.2.1 shows that it is easy to compute the determinant of an upper or lower triangular matrix. Recall from Chapter 2 that *any* matrix can be reduced to row-echelon form by a sequence of elementary row operations. In the case of an $n \times n$ matrix, any row-echelon form will be upper triangular. Theorem 3.2.1 suggests, therefore, that we should consider how elementary row operations performed on a matrix A alter the value of $\det(A)$.

Elementary Row Operations and Determinants

Let A be an $n \times n$ matrix.

P1. If B is the matrix obtained by permuting two rows of A, then

$$\det(B) = -\det(A).$$

P2. If B is the matrix obtained by multiplying one row of A by any[2] scalar k, then

$$\det(B) = k\,\det(A).$$

P3. If B is the matrix obtained by adding a multiple of any row of A to a different row of A, then

$$\det(B) = \det(A).$$

The proofs of these properties are given at the end of this section.

Remark The main use of P2 is that it enables us to factor a common multiple of the entries of a particular row out of the determinant. For example, if

$$A = \begin{bmatrix} -1 & 4 \\ 3 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -5 & 20 \\ 3 & -2 \end{bmatrix},$$

where B is obtained from A by multiplying the first row of A by 5, then we have

$$\det(B) = 5\det(A) = 5[(-1)(-2) - (3)(4)] = 5(-10) = -50.$$

We now illustrate how the foregoing properties P1–P3, together with Theorem 3.2.1, can be used to evaluate a determinant. The basic idea is the same as that for Gaussian elimination. We use elementary row operations to reduce the determinant to upper triangular form and then use Theorem 3.2.1 to evaluate the resulting determinant.

Warning: When using the properties P1–P3 to simplify a determinant, one must remember to take account of any change that arises in the value of the determinant from the operations that have been performed on it.

Example 3.2.3 Evaluate $\begin{vmatrix} 2 & -1 & 3 & 7 \\ 1 & -2 & 4 & 3 \\ 3 & 4 & 2 & -1 \\ 2 & -2 & 8 & -4 \end{vmatrix}$.

[2] This statement is even true if $k = 0$.

Solution: We have

$$\begin{vmatrix} 2 & -1 & 3 & 7 \\ 1 & -2 & 4 & 3 \\ 3 & 4 & 2 & -1 \\ 2 & -2 & 8 & -4 \end{vmatrix} \overset{1}{=} 2 \begin{vmatrix} 2 & -1 & 3 & 7 \\ 1 & -2 & 4 & 3 \\ 3 & 4 & 2 & -1 \\ 1 & -1 & 4 & -2 \end{vmatrix} \overset{2}{=} -2 \begin{vmatrix} 1 & -2 & 4 & 3 \\ 2 & -1 & 3 & 7 \\ 3 & 4 & 2 & -1 \\ 1 & -1 & 4 & -2 \end{vmatrix} \overset{3}{=} -2 \begin{vmatrix} 1 & -2 & 4 & 3 \\ 0 & 3 & -5 & 1 \\ 0 & 10 & -10 & -10 \\ 0 & 1 & 0 & -5 \end{vmatrix}$$

$$\overset{4}{=} 2 \begin{vmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 10 & -10 & -10 \\ 0 & 3 & -5 & 1 \end{vmatrix} \overset{5}{=} 20 \begin{vmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -5 & 1 \end{vmatrix} \overset{6}{=} 20 \begin{vmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & -5 & 16 \end{vmatrix}$$

$$\overset{7}{=} 20 \begin{vmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & -4 \end{vmatrix} = 80.$$

1. $M_4(\frac{1}{2})$	**2.** P_{12}	**3.** $A_{12}(-2)$, $A_{13}(-3)$, $A_{14}(-1)$	**4.** P_{24}
5. $M_3(\frac{1}{10})$	**6.** $A_{23}(-1)$, $A_{24}(-3)$	**7.** $A_{34}(-5)$	

□

Theoretical Results for $n \times n$ Matrices and $n \times n$ Linear Systems

In Section 2.8, we established several conditions on an $n \times n$ matrix A that are equivalent to saying that A is invertible. At this point, we are ready to give one additional characterization of invertible matrices in terms of determinants.

Theorem 3.2.4 Let A be an $n \times n$ matrix with real elements. The following conditions on A are equivalent.

(a) A is invertible.

(g) $\det(A) \neq 0$.

Proof Let A^* denote the reduced row-echelon form of A. Recall from Chapter 2 that A is invertible if and only if $A^* = I_n$. Since A^* is obtained from A by performing a sequence of elementary row operations, properties P1–P3 of determinants imply that $\det(A)$ is just a *nonzero* multiple of $\det(A^*)$. If A is invertible, then $\det(A^*) = \det(I_n) = 1$, so that $\det(A)$ is nonzero.

Conversely, if $\det(A) \neq 0$, then $\det(A^*) \neq 0$. This implies that $A^* = I_n$, hence A is invertible. ■

According to Theorem 2.5.9 in the previous chapter, any linear system $A\mathbf{x} = \mathbf{b}$ has either no solution, exactly one solution, or infinitely many solutions. Recall from the Invertible Matrix Theorem that the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$ if and only if A is invertible. Thus, for an $n \times n$ linear system, Theorem 3.2.4 tells us that, for each $\mathbf{b}$ in $\mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ if and only if $\det(A) \neq 0$.

Next, we consider the homogeneous $n \times n$ linear system $A\mathbf{x} = \mathbf{0}$.

Corollary 3.2.5 The homogeneous $n \times n$ linear system $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions if and only if $\det(A) = 0$, and has only the trivial solution if and only if $\det(A) \neq 0$.

Proof The system $A\mathbf{x} = \mathbf{0}$ clearly has the trivial solution $\mathbf{x} = \mathbf{0}$ under any circumstances. By our remarks above, this must be the unique solution if and only if $\det(A) \neq 0$. The only other possibility, which occurs if and only if $\det(A) = 0$, is that the system has infinitely many solutions. ■

Remark The preceding corollary is *very* important, since we are often interested only in determining the solution properties of a homogeneous linear system and not actually in finding the solutions themselves. We will refer back to this corollary on many occasions throughout the remainder of the text.

Example 3.2.6 Verify that the matrix

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 4 & -2 \\ 3 & 5 & 7 \end{bmatrix}$$

is invertible. What can be concluded about the solution to $A\mathbf{x} = \mathbf{0}$?

Solution: It is easily shown that $\det(A) = 52 \neq 0$. Consequently, A is invertible. It follows from Corollary 3.2.5 that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $(0, 0, 0)$. □

Example 3.2.7 Verify that the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -3 & 0 & -3 \end{bmatrix}$$

is not invertible and determine a set of real solutions to the system $A\mathbf{x} = \mathbf{0}$.

Solution: By the row operation $\mathrm{A}_{13}(3)$, we see that A is row equivalent to the upper triangular matrix

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 3.2.1, $\det(B) = 0$, and hence B and A are not invertible. We illustrate Corollary 3.2.5 by finding an infinite number of solutions (x_1, x_2, x_3) to $A\mathbf{x} = \mathbf{0}$. Working with the upper triangular matrix B, we may set $x_3 = t$, a free parameter. The second row of the matrix system requires that $x_2 = 0$ and the first row requires that $x_1 + x_3 = 0$, so $x_1 = -x_3 = -t$. Hence, the set of solutions is $\{(-t, 0, t) : t \in \mathbb{R}\}$. □

Further Properties of Determinants

In addition to elementary row operations, the following properties can also be useful in evaluating determinants.
Let A and B be $n \times n$ matrices.

P4. $\det(A^T) = \det(A)$.

P5. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ denote the row vectors of A. If the ith row vector of A is the sum of two row vectors, say $\mathbf{a}_i = \mathbf{b}_i + \mathbf{c}_i$, then $\det(A) = \det(B) + \det(C)$, where

$$B = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{i-1} \\ \mathbf{b}_i \\ \mathbf{a}_{i+1} \\ \vdots \\ \mathbf{a}_n \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{i-1} \\ \mathbf{c}_i \\ \mathbf{a}_{i+1} \\ \vdots \\ \mathbf{a}_n \end{bmatrix}.$$

The corresponding property is also true for columns.

P6. If A has a row (or column) of zeros, then $\det(A) = 0$.

P7. If two rows (or columns) of A are the same, then $\det(A) = 0$.

P8. $\det(AB) = \det(A)\det(B)$.

The proofs of these properties are given at the end of the section. The main importance of P4 is the implication that any results regarding determinants that hold for the rows of a matrix also hold for the columns of a matrix. In particular, the properties P1–P3 regarding the effects that elementary row operations have on the determinant can be translated to corresponding statements on the effects that "elementary column operations" have on the determinant. We will use the notations

$$\mathrm{CP}_{ij}, \qquad \mathrm{CM}_i(k), \qquad \text{and} \qquad \mathrm{CA}_{ij}(k)$$

to denote the three types of elementary column operations.

Example 3.2.8 Use only column operations to evaluate

$$\begin{vmatrix} 3 & 6 & -1 & 2 \\ 6 & 10 & 3 & 4 \\ 9 & 20 & 5 & 4 \\ 15 & 34 & 3 & 8 \end{vmatrix}.$$

Solution: We have

$$\begin{vmatrix} 3 & 6 & -1 & 2 \\ 6 & 10 & 3 & 4 \\ 9 & 20 & 5 & 4 \\ 15 & 34 & 3 & 8 \end{vmatrix} \overset{1}{=} 3 \cdot 2^2 \begin{vmatrix} 1 & 3 & -1 & 1 \\ 2 & 5 & 3 & 2 \\ 3 & 10 & 5 & 2 \\ 5 & 17 & 3 & 4 \end{vmatrix} \overset{2}{=} 12 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 5 & 0 \\ 3 & 1 & 8 & -1 \\ 5 & 2 & 8 & -1 \end{vmatrix} \overset{3}{=} 12 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 1 & 13 & -1 \\ 5 & 2 & 18 & -1 \end{vmatrix}$$

$$\overset{4}{=} 12 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 1 & 13 & 0 \\ 5 & 2 & 18 & \frac{5}{13} \end{vmatrix} = 12(-5) = -60,$$

where we have once more used Theorem 3.2.1.

1. $\mathrm{CM}_1(\frac{1}{3})$, $\mathrm{CM}_2(\frac{1}{2})$, $\mathrm{CM}_4(\frac{1}{2})$ **2.** $\mathrm{CA}_{12}(-3)$, $\mathrm{CA}_{13}(1)$, $\mathrm{CA}_{14}(-1)$
3. $\mathrm{CA}_{23}(5)$ **4.** $\mathrm{CA}_{34}(\frac{1}{13})$

□

The property that often gives the most difficulty is P5. We explicitly illustrate its use with an example.

Example 3.2.9 Use property P5 to express

$$\begin{vmatrix} a_1+b_1 & c_1+d_1 \\ a_2+b_2 & c_2+d_2 \end{vmatrix}$$

as a sum of four determinants.

Solution: Applying P5 to row 1 yields:

$$\begin{vmatrix} a_1+b_1 & c_1+d_1 \\ a_2+b_2 & c_2+d_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2+b_2 & c_2+d_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ a_2+b_2 & c_2+d_2 \end{vmatrix}.$$

Now we apply P5 to row 2 of both of the determinants on the right-hand side to obtain

$$\begin{vmatrix} a_1+b_1 & c_1+d_1 \\ a_2+b_2 & c_2+d_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 \\ b_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}.$$

Notice that we could also have applied P5 to the columns of the given determinant. □

Warning In view of P5, it may be tempting to believe that if A, B, and C are $n \times n$ matrices such that $A = B + C$, then $\det(A) = \det(B) + \det(C)$. This is not true! Examples abound to show the failure of this equation. For instance, if we take $B = I_2$ and $C = -I_2$, then $\det(A) = \det(0_2) = 0$, while $\det(B) = \det(C) = 1$. Thus, $\det(B) + \det(C) = 1 + 1 = 2 \neq 0$.

Next, we supply some examples of the last two properties, P7 and P8.

Example 3.2.10 Evaluate

(a) $\begin{vmatrix} 1 & 2 & -3 & 1 \\ -2 & 4 & 6 & 2 \\ -3 & -6 & 9 & 3 \\ 2 & 11 & -6 & 4 \end{vmatrix}$.

(b) $\begin{vmatrix} 2-4x & -4 & 2 \\ 5+3x & 3 & -3 \\ 1-2x & -2 & 1 \end{vmatrix}$.

Solution:

(a) We have

$$\begin{vmatrix} 1 & 2 & -3 & 1 \\ -2 & 4 & 6 & 2 \\ -3 & -6 & 9 & 3 \\ 2 & 11 & -6 & 4 \end{vmatrix} \overset{1}{=} -3\begin{vmatrix} 1 & 2 & 1 & 1 \\ -2 & 4 & -2 & 2 \\ -3 & -6 & -3 & 3 \\ 2 & 11 & 2 & 4 \end{vmatrix} = 0,$$

since the first and third columns of the latter matrix are identical (see P7).

$\boxed{\mathbf{1}.\ \mathrm{CM}_3(-\tfrac{1}{3})}$

(b) Applying P5 to the first column, we have

$$\begin{vmatrix} 2-4x & -4 & 2 \\ 5+3x & 3 & -3 \\ 1-2x & -2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -4 & 2 \\ 5 & 3 & -3 \\ 1 & -2 & 1 \end{vmatrix} + \begin{vmatrix} -4x & -4 & 2 \\ 3x & 3 & -3 \\ -2x & -2 & 1 \end{vmatrix}$$
$$= 2\begin{vmatrix} 1 & -2 & 1 \\ 5 & 3 & -3 \\ 1 & -2 & 1 \end{vmatrix} + x\begin{vmatrix} -4 & -4 & 2 \\ 3 & 3 & -3 \\ -2 & -2 & 1 \end{vmatrix} = 0 + 0 = 0,$$

since the first and third rows of the first matrix agree, and the first and second columns of the second matrix agree. □

Example 3.2.11 If

$$A = \begin{bmatrix} \sin\phi & \cos\phi \\ -\cos\phi & \sin\phi \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

show that $\det(AB) = 1$.

Solution: Using P8, we have

$$\det(AB) = \det(A)\det(B) = (\sin^2\phi + \cos^2\phi)(\cos^2\theta + \sin^2\theta) = 1 \cdot 1 = 1. \quad \square$$

Example 3.2.12 Find all x satisfying

$$\begin{vmatrix} x^2 & x & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 0.$$

Solution: If we expanded this determinant according to Definition 3.1.8 (or using the schematic in Figure 3.1.1), then we would have a quadratic equation in x. Thus, there are at most two distinct values of x that satisfy the equation. By inspection, the determinant vanishes when $x = 1$ (since the first two rows of the matrix coincide in this case), and it vanishes when $x = 2$ (since the first and third rows of the matrix coincide in this case). Consequently, the two values of x satisfying the given equation are $x = 1$ and $x = 2$. □

Proofs of the Properties of Determinants

We now prove the properties P1–P8.

Proof of P1: Let B be the matrix obtained by interchanging row r with row s in A. Then the elements of B are related to those of A as follows:

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i \neq r, s, \\ a_{sj} & \text{if } i = r, \\ a_{rj} & \text{if } i = s. \end{cases}$$

Thus, from Definition 3.1.8,

$$\begin{aligned}\det(B) &= \sum \sigma(p_1, p_2, \cdots, p_r, \cdots, p_s, \cdots, p_n) b_{1p_1} b_{2p_2} \cdots b_{rp_r} \cdots b_{sp_s} \cdots b_{np_n} \\ &= \sum \sigma(p_1, p_2, \cdots, p_r, \cdots, p_s, \cdots, p_n) a_{1p_1} a_{2p_2} \cdots a_{sp_r} \cdots a_{rp_s} \cdots a_{np_n}.\end{aligned}$$

Interchanging p_r and p_s in $\sigma(p_1, p_2, \ldots, p_r, \ldots, p_s, \ldots, p_n)$ and recalling from Theorem 3.1.7 that such an interchange has the effect of changing the parity of the permutation, we obtain

$$\det(B) = -\sum \sigma(p_1, p_2, \cdots, p_s, \cdots, p_r, \cdots, p_n) a_{1p_1} a_{2p_2} \cdots a_{rp_s} \cdots a_{sp_r} \cdots a_{np_n},$$

where we have also rearranged the terms so that the row indices are in their natural increasing order. The sum on the right-hand side of this equation is just $\det(A)$, so that

$$\det(B) = -\det(A). \qquad \blacksquare$$

Proof of P2: Let B be the matrix obtained by multiplying the ith row of A through by any scalar k. Then $b_{ij} = ka_{ij}$ for each j. Then

$$\begin{aligned}\det(B) &= \sum \sigma(p_1, p_2, \cdots, p_n) b_{1p_1} b_{2p_2} \cdots b_{np_n} \\ &= \sum \sigma(p_1, p_2, \cdots, p_n) a_{1p_1} a_{2p_2} \cdots (ka_{ip_i}) \cdots a_{np_n} = k \det(A).\end{aligned} \qquad \blacksquare$$

We prove properties P5 and P7 next, since they simplify the proof of P3.

Proof of P5: The elements of A are

$$a_{kj} = \begin{cases} a_{kj}, & \text{if } k \neq i, \\ b_{ij} + c_{ij}, & \text{if } k = i. \end{cases}$$

Thus, from Definition 3.1.8,

$$\begin{aligned}\det(A) &= \sum \sigma(p_1, p_2, \cdots, p_n) a_{1p_1} a_{2p_2} \cdots a_{np_n} \\ &= \sum \sigma(p_1, p_2, \cdots, p_n) a_{1p_1} a_{2p_2} \cdots a_{i-1p_{i-1}} (b_{ip_i} + c_{ip_i}) a_{i+1p_{i+1}} \cdots a_{np_n} \\ &= \sum \sigma(p_1, p_2, \cdots, p_n) a_{1p_1} a_{2p_2} \cdots a_{i-1p_{i-1}} b_{ip_i} a_{i+1p_{i+1}} \cdots a_{np_n} \\ &+ \sum \sigma(p_1, p_2, \cdots, p_n) a_{1p_1} a_{2p_2} \cdots a_{i-1p_{i-1}} c_{ip_i} a_{i+1p_{i+1}} \cdots a_{np_n} \\ &= \det(B) + \det(C).\end{aligned} \qquad \blacksquare$$

Proof of P7: Suppose rows i and j in A are the same. Then if we interchange these rows, the matrix, and hence its determinant, are unaltered. However, according to P1, the determinant of the resulting matrix is $-\det(A)$. Therefore,

$$\det(A) = -\det(A),$$

which implies that

$$\det(A) = 0. \qquad \blacksquare$$

Proof of P3: Let $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]^T$, and let B be the matrix obtained from A when k times row j of A is added to row i of A. Then

$$B = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_i + k\mathbf{a}_j, \ldots, \mathbf{a}_n]^T$$

so that, using P5,

$$\begin{aligned}\det(B) &= \det([\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_i + k\mathbf{a}_j, \dots, \mathbf{a}_n]^T) \\ &= \det([\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]^T) + \det([\mathbf{a}_1, \mathbf{a}_2, \dots, k\mathbf{a}_j, \dots, \mathbf{a}_n]^T).\end{aligned}$$

By P2, we can factor out k from row i of the second determinant on the right-hand side. If we do this, it follows that row i and row j of the resulting determinant are the same, and so, from P7, the value of the second determinant is zero. Thus,

$$\det(B) = \det([\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]^T) = \det(A),$$

as required. ■

Proof of P4: Using Definition 3.1.8, we have

$$\det(A^T) = \sum \sigma(p_1, p_2, \dots, p_n) a_{p_1 1} a_{p_2 2} a_{p_3 3} \cdots a_{p_n n}. \tag{3.2.2}$$

Since $(p_1, p_2, \dots, p_n)$ is a permutation of $1, 2, \dots, n$, it follows that, by rearranging terms,

$$a_{p_1 1} a_{p_2 2} a_{p_3 3} \cdots a_{p_n n} = a_{1q_1} a_{2q_2} a_{3q_3} \cdots a_{nq_n}, \tag{3.2.3}$$

for appropriate values of $q_1, q_2, \dots, q_n$. Furthermore,

$$\begin{aligned}N(p_1, \dots, p_n) &= \#\text{ of interchanges in changing } (1, 2, \dots, n) \text{ to } (p_1, p_2, \dots, p_n) \\ &= \#\text{ of interchanges in changing } (p_1, p_2, \dots, p_n) \text{ to } (1, 2, \dots, n)\end{aligned}$$

and by (3.2.3), this number is

$$\begin{aligned}&= \#\text{ of interchanges in changing } (1, 2, \dots, n) \text{ to } (q_1, q_2, \dots, q_n) \\ &= N(q_1, \dots, q_n).\end{aligned}$$

Thus,

$$\sigma(p_1, p_2, \dots, p_n) = \sigma(q_1, q_2, \dots, q_n). \tag{3.2.4}$$

Substituting Equations (3.2.3) and (3.2.4) into Equation (3.2.2), we have

$$\begin{aligned}\det(A^T) &= \sum \sigma(q_1, q_2, \dots, q_n) a_{1q_1} a_{2q_2} a_{3q_3} \cdots a_{nq_n} \\ &= \det(A).\end{aligned}$$

■

Proof of P6: Since each term $\sigma(p_1, p_2, \dots, p_n) a_{1p_1} a_{2p_2} \cdots a_{np_n}$ in the formula for $\det(A)$ contains a factor from the row (or column) of zeros, each such term is zero. Thus, $\det(A) = 0$. ■

Proof of P8: Let E denote an elementary matrix. We leave it as an exercise (Problem 51) to verify that

$$\det(E) = \begin{cases} -1, & \text{if } E \text{ permutes rows,} \\ +1, & \text{if } E \text{ adds a multiple of one row to another row,} \\ k, & \text{if } E \text{ scales a row by } k. \end{cases}$$

It then follows from properties P1–P3 that in each case

$$\det(EA) = \det(E)\det(A). \tag{3.2.5}$$

Now consider a general product AB. We need to distinguish two cases.

Case 1: If A is not invertible, then from Corollary 2.6.12, so is AB. Consequently, applying Theorem 3.2.4,

$$\det(AB) = 0 = \det(A)\det(B).$$

Case 2: If A is invertible, then from Section 2.7, we know that it can be expressed as the product of elementary matrices, say, $A = E_1E_2\cdots E_r$. Hence, repeatedly applying (3.2.5) gives

$$\begin{aligned}\det(AB) &= \det(E_1E_2\cdots E_rB) = \det(E_1)\det(E_2\cdots E_rB)\\ &= \det(E_1)\det(E_2)\cdots\det(E_r)\det(B)\\ &= \det(E_1E_2\cdots E_r)\det(B) = \det(A)\det(B).\end{aligned}$$ ■

Exercises for 3.2

Skills

- Be able to compute the determinant of an upper or lower triangular matrix "at a glance" (Theorem 3.2.1).
- Know the effects that elementary row operations have on the determinant of a matrix.
- Likewise, be comfortable with the effects that column operations have on the determinant of a matrix.
- Be able to use the determinant to decide if a matrix is invertible (Theorem 3.2.4).
- Know how the determinant is affected by matrix multiplication and by matrix transpose.

True-False Review

For Questions 1–6, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. If each element of an $n \times n$ matrix is doubled, then the determinant of the matrix also doubles.

2. Multiplying a row of an $n \times n$ matrix through by a scalar c has the same effect on the determinant as multiplying a column of the matrix through by c.

3. If A is an $n \times n$ matrix, then $\det(A^5) = (\det\ A)^5$.

4. If A is a real $n \times n$ matrix, then $\det(A^2)$ cannot be negative.

5. The matrix $\begin{bmatrix} x^2 & x \\ y^2 & y \end{bmatrix}$ is not invertible if and only if $x = 0$ or $y = 0$.

6. If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$.

Problems

For Problems 1–12, reduce the given determinant to upper triangular form and then evaluate.

1. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & -5 & 2 \end{vmatrix}$.

2. $\begin{vmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ -2 & 1 & 4 \end{vmatrix}$.

3. $\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 6 \\ 4 & 1 & 12 \end{vmatrix}$.

4. $\begin{vmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix}$.

5. $\begin{vmatrix} 3 & 7 & 1 \\ 5 & 9 & -6 \\ 2 & 1 & 3 \end{vmatrix}$.

6. $\begin{vmatrix} 1 & -1 & 2 & 4 \\ 3 & 1 & 2 & 4 \\ -1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 2 \end{vmatrix}$.

7. $\begin{vmatrix} 2 & 32 & 1 & 4 \\ 26 & 104 & 26 & -13 \\ 2 & 56 & 2 & 7 \\ 1 & 40 & 1 & 5 \end{vmatrix}$.

8. $\begin{vmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix}$.

9. $\begin{vmatrix} 2 & 1 & 3 & 5 \\ 3 & 0 & 1 & 2 \\ 4 & 1 & 4 & 3 \\ 5 & 2 & 5 & 3 \end{vmatrix}$.

10. $\begin{vmatrix} 2 & -1 & 3 & 4 \\ 7 & 1 & 2 & 3 \\ -2 & 4 & 8 & 6 \\ 6 & -6 & 18 & -24 \end{vmatrix}$.

11. $\begin{vmatrix} 7 & -1 & 3 & 4 \\ 14 & 2 & 4 & 6 \\ 21 & 1 & 3 & 4 \\ -7 & 4 & 5 & 8 \end{vmatrix}$.

12. $\begin{vmatrix} 3 & 7 & 1 & 2 & 3 \\ 1 & 1 & -1 & 0 & 1 \\ 4 & 8 & -1 & 6 & 6 \\ 3 & 7 & 0 & 9 & 4 \\ 8 & 16 & -1 & 8 & 12 \end{vmatrix}$.

For Problems 13–19, use Theorem 3.2.4 to determine whether the given matrix is invertible or not.

13. $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$.

14. $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

15. $\begin{bmatrix} 2 & 6 & -1 \\ 3 & 5 & 1 \\ 2 & 0 & 1 \end{bmatrix}$.

16. $\begin{bmatrix} -1 & 2 & 3 \\ 5 & -2 & 1 \\ 8 & -2 & 5 \end{bmatrix}$.

17. $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 3 & -2 & 1 & 4 \\ 2 & 1 & 6 & 2 \\ 1 & -3 & 4 & 0 \end{bmatrix}$.

18. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$.

19. $\begin{bmatrix} 1 & 2 & -3 & 5 \\ -1 & 2 & -3 & 6 \\ 2 & 3 & -1 & 4 \\ 1 & -2 & 3 & -6 \end{bmatrix}$.

20. Determine all values of the constant k for which the given system has a unique solution

$$\begin{aligned} x_1 + kx_2 &= b_1, \\ kx_1 + 4x_2 &= b_2. \end{aligned}$$

21. Determine all values of the constant k for which the given system has an infinite number of solutions.

$$\begin{aligned} x_1 + 2x_2 + kx_3 &= 0, \\ 2x_1 - kx_2 + x_3 &= 0, \\ 3x_1 + 6x_2 + x_3 &= 0. \end{aligned}$$

22. Determine all values of k for which the given system has an infinite number of solutions.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= kx_1, \\ 2x_1 + x_2 + x_3 &= kx_2, \\ x_1 + x_2 + 2x_3 &= kx_3. \end{aligned}$$

23. Determine all values of k for which the given system has a unique solution.

$$\begin{aligned} x_1 + kx_2 &= 2, \\ kx_1 + x_2 + x_3 &= 1, \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

24. If

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix},$$

find $\det(A)$, and use properties of determinants to find $\det(A^T)$ and $\det(-2A)$.

25. If

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix},$$

evaluate $\det(AB)$ and verify P8.

26. If

$$A = \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix} \text{ and } B = \begin{bmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{bmatrix},$$

evaluate $\det(AB)$.

For Problems 27–29, use properties of determinants to show that $\det(A) = 0$ for the given matrix A.

27. $A = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & -1 \\ 9 & 6 & 2 \end{bmatrix}$.

28. $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -1 & 7 \\ 3 & 1 & 13 \end{bmatrix}$.

29. $A = \begin{bmatrix} 1+3a & 1 & 3 \\ 1+2a & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix}$.

For Problems 30–32, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and assume $\det(A) = 1$. Find $\det(B)$.

30. $B = \begin{bmatrix} 3c & 3d \\ 4a & 4b \end{bmatrix}$.

31. $B = \begin{bmatrix} -2a & -2c \\ 3a+b & 3c+d \end{bmatrix}$.

32. $B = \begin{bmatrix} -b & -a \\ d-4b & c-4a \end{bmatrix}$.

For Problems 33–35, let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and assume $\det(A) = -6$. Find $\det(B)$.

33. $B = \begin{bmatrix} -4d & -4e & -4f \\ g+5a & h+5b & i+5c \\ a & b & c \end{bmatrix}$.

34. $B = \begin{bmatrix} d & e & f \\ -3a & -3b & -3c \\ g-4d & h-4e & i-4f \end{bmatrix}$.

35. $B = \begin{bmatrix} 2a & 2d & 2g \\ b-c & e-f & h-i \\ c-a & f-d & i-g \end{bmatrix}$.

For Problems 36–40, let A and B be 4×4 matrices such that $\det(A) = 5$ and $\det(B) = 3$. Compute the determinant of the given matrix.

36. AB^T.

37. A^2B^5.

38. $(A^{-1}B^2)^3$.

39. $((2B)^{-1}(AB)^T)$.

40. $(5A)(2B)$.

41. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix}.$$

(a) In terms of k, find the volume of the parallelepiped determined by the row vectors of the matrix A.

(b) Does your answer to (a) change if we instead consider the volume of the parallelepiped determined by the *column* vectors of the matrix A? Why or why not?

(c) For what value(s) of k, if any, is A invertible?

42. Without expanding the determinant, determine all values of x for which $\det(A) = 0$ if

$$A = \begin{bmatrix} 1 & -1 & x \\ 2 & 1 & x^2 \\ 4 & -1 & x^3 \end{bmatrix}.$$

43. Use *only* properties P5, P1, and P2 to show that

$$\begin{vmatrix} \alpha x - \beta y & \beta x - \alpha y \\ \beta x + \alpha y & \alpha x + \beta y \end{vmatrix} = (x^2 + y^2)\begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix}.$$

44. Use *only* properties P5, P1, and P2 to find the value of $\alpha\beta\gamma$ such that

$$\begin{vmatrix} a_1 + \beta b_1 & b_1 + \gamma c_1 & c_1 + \alpha a_1 \\ a_2 + \beta b_2 & b_2 + \gamma c_2 & c_2 + \alpha a_2 \\ a_3 + \beta b_3 & b_3 + \gamma c_3 & c_3 + \alpha a_3 \end{vmatrix} = 0$$

for all values of a_i, b_i, c_i.

45. Use *only* properties P3 and P7 to prove property P6.

46. An $n \times n$ matrix A that satisfies $A^T = A^{-1}$ is called an **orthogonal matrix**. Show that if A is an orthogonal matrix, then $\det(A) = \pm 1$.

47. (a) Use the definition of a determinant to prove that if A is an $n \times n$ lower triangular matrix, then

$$\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

(b) Evaluate the following determinant by first reducing it to lower triangular form and then using the result from (a):

$$\begin{vmatrix} 2 & -1 & 3 & 5 \\ 1 & 2 & 2 & 1 \\ 3 & 0 & 1 & 4 \\ 1 & 2 & 0 & 1 \end{vmatrix}.$$

48. Use determinants to prove that if A is invertible and B and C are matrices with $AB = AC$, then $B = C$.

49. If A and S are $n \times n$ matrices with S invertible, show that $\det(S^{-1}AS) = \det(A)$. [**Hint:** Since $S^{-1}S = I_n$, how are $\det(S^{-1})$ and $\det(S)$ related?]

50. If $\det(A^3) = 0$, is it possible for A to be invertible? Justify your answer.

51. Let E be an elementary matrix. Verify the formula for $\det(E)$ given in the text at the beginning of the proof of P8.

52. Show that

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

represents the equation of the straight line through the distinct points (x_1, y_1) and (x_2, y_2).

53. Without expanding the determinant, show that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y - z)(z - x)(x - y).$$

54. If A is an $n \times n$ *skew-symmetric* matrix and n is odd, prove that $\det(A) = 0$.

55. Let $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ be an $n \times n$ matrix, and let $\mathbf{b} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n$, where $c_1, c_2, \ldots, c_n$ are constants. If B_k denotes the matrix obtained from A by replacing the kth column vector by $\mathbf{b}$, prove that

$$\det(B_k) = c_k \det(A), \qquad k = 1, 2, \ldots, n.$$

56. ◇ Let A be the general 4×4 matrix.

(a) Verify property P1 of determinants in the case when the first two rows of A are permuted.

(b) Verify property P2 of determinants in the case when row 1 of A is divided by k.

(c) Verify property P3 of determinants in the case when k times row 2 is added to row 1.

57. ◇ For a randomly generated 5×5 matrix, verify that $\det(A^T) = \det(A)$.

58. ◇ Determine all values of a for which

$$\begin{bmatrix} 1 & 2 & 3 & 4 & a \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ a & 4 & 3 & 2 & 1 \end{bmatrix}$$

is invertible.

59. ◇ If

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 3 & 2 & 1 \\ 3 & 4 & -1 \end{bmatrix},$$

determine all values of the constant k for which the linear system $(A - kI_3)\mathbf{x} = \mathbf{0}$ has an infinite number of solutions, and find the corresponding solutions.

60. ◇ Use the determinant to show that

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

is invertible, and use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = [3, 7, 1, -4]^T$.

3.3 Cofactor Expansions

We now obtain an alternative method for evaluating determinants. The basic idea is that we can reduce a determinant of order n to a sum of determinants of order $n-1$. Continuing in this manner, it is possible to express any determinant as a sum of determinants of order 2. This method is the one most frequently used to evaluate a determinant by hand, although the procedure introduced in the previous section whereby we use elementary row operations to reduce the matrix to upper triangular form involves less work in general. When A is invertible, the technique we derive leads to formulas for both A^{-1} and the unique solution to $A\mathbf{x} = \mathbf{b}$. We first require two preliminary definitions.

DEFINITION 3.3.1

Let A be an $n \times n$ matrix. The **minor**, M_{ij}, of the element a_{ij}, is the determinant of the matrix obtained by deleting the ith row vector and jth column vector of A.