49. If A and S are $n \times n$ matrices with S invertible, show that $\det(S^{-1}AS) = \det(A)$. [**Hint:** Since $S^{-1}S = I_n$, how are $\det(S^{-1})$ and $\det(S)$ related?]

50. If $\det(A^3) = 0$, is it possible for A to be invertible? Justify your answer.

51. Let E be an elementary matrix. Verify the formula for $\det(E)$ given in the text at the beginning of the proof of P8.

52. Show that

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

represents the equation of the straight line through the distinct points (x_1, y_1) and (x_2, y_2).

53. Without expanding the determinant, show that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y - z)(z - x)(x - y).$$

54. If A is an $n \times n$ *skew-symmetric* matrix and n is odd, prove that $\det(A) = 0$.

55. Let $A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]$ be an $n \times n$ matrix, and let $\mathbf{b} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n$, where $c_1, c_2, \dots, c_n$ are constants. If B_k denotes the matrix obtained from A by replacing the kth column vector by $\mathbf{b}$, prove that

$$\det(B_k) = c_k \det(A), \qquad k = 1, 2, \dots, n.$$

56. ◇ Let A be the general 4×4 matrix.

(a) Verify property P1 of determinants in the case when the first two rows of A are permuted.

(b) Verify property P2 of determinants in the case when row 1 of A is divided by k.

(c) Verify property P3 of determinants in the case when k times row 2 is added to row 1.

57. ◇ For a randomly generated 5×5 matrix, verify that $\det(A^T) = \det(A)$.

58. ◇ Determine all values of a for which

$$\begin{bmatrix} 1 & 2 & 3 & 4 & a \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ a & 4 & 3 & 2 & 1 \end{bmatrix}$$

is invertible.

59. ◇ If

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 3 & 2 & 1 \\ 3 & 4 & -1 \end{bmatrix},$$

determine all values of the constant k for which the linear system $(A - kI_3)\mathbf{x} = \mathbf{0}$ has an infinite number of solutions, and find the corresponding solutions.

60. ◇ Use the determinant to show that

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

is invertible, and use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = [3, 7, 1, -4]^T$.

3.3   Cofactor Expansions

We now obtain an alternative method for evaluating determinants. The basic idea is that we can reduce a determinant of order n to a sum of determinants of order $n - 1$. Continuing in this manner, it is possible to express any determinant as a sum of determinants of order 2. This method is the one most frequently used to evaluate a determinant by hand, although the procedure introduced in the previous section whereby we use elementary row operations to reduce the matrix to upper triangular form involves less work in general. When A is invertible, the technique we derive leads to formulas for both A^{-1} and the unique solution to $A\mathbf{x} = \mathbf{b}$. We first require two preliminary definitions.

DEFINITION   3.3.1

Let A be an $n \times n$ matrix. The **minor**, M_{ij}, of the element a_{ij}, is the determinant of the matrix obtained by deleting the ith row vector and jth column vector of A.

Remark   Notice that if A is an $n \times n$ matrix, then M_{ij} is a determinant of order $n - 1$. By convention, if $n = 1$, we define the "empty" determinant M_{11} to be 1.

Example 3.3.2   If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then, for example,

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \quad \text{and} \quad M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}. \qquad \square$$

Example 3.3.3   Determine the minors M_{11}, M_{23}, and M_{31} for

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 3 & 1 & 5 \end{bmatrix}.$$

Solution: Using Definition 3.3.1, we have

$$M_{11} = \begin{vmatrix} 4 & -2 \\ 1 & 5 \end{vmatrix} = 22, \qquad M_{23} = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1, \qquad M_{31} = \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = -14. \qquad \square$$

DEFINITION   3.3.4

Let A be an $n \times n$ matrix. The **cofactor**, C_{ij}, of the element a_{ij}, is defined by

$$C_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the minor of a_{ij}.

From Definition 3.3.4, we see that the cofactor of a_{ij} and the minor of a_{ij} are the same if $i + j$ is even, and they differ by a minus sign if $i + j$ is odd. The appropriate sign in the cofactor C_{ij} is easy to remember, since it alternates in the following manner:

$$\begin{vmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}.$$

Example 3.3.5   Determine the cofactors C_{11}, C_{23}, and C_{31} for the matrix in Example 3.3.3.

Solution: We have already obtained the minors M_{11}, M_{23}, and M_{31} in Example 3.3.3, so it follows that

$$C_{11} = +M_{11} = 22, \qquad C_{23} = -M_{23} = 1, \qquad C_{31} = +M_{31} = -14. \qquad \square$$

Example 3.3.6   If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, verify that $\det(A) = a_{11}C_{11} + a_{12}C_{12}$.

Solution: In this case,

$$C_{11} = +\det[a_{22}] = a_{22}, \qquad C_{12} = -\det[a_{12}] = -a_{12},$$

so that

$$a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} + a_{12}(-a_{21}) = \det(A). \qquad \square$$

The preceding example is a special case of the following important theorem.

Theorem 3.3.7   **(Cofactor Expansion Theorem)**

Let A be an $n \times n$ matrix. If we multiply the elements in any row (or column) of A by their cofactors, then the sum of the resulting products is $\det(A)$. Thus,

1. If we expand along row i,

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}.$$

2. If we expand along column j,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{k=1}^{n} a_{kj}C_{kj}.$$

The expressions for $\det(A)$ appearing in this theorem are known as **cofactor expansions**. Notice that a cofactor expansion can be formed along *any* row or column of A. Regardless of the chosen row or column, the cofactor expansion will always yield the determinant of A. However, sometimes the calculation is simpler if the row or column of expansion is wisely chosen. We will illustrate this in the examples below. The proof of the Cofactor Expansion Theorem will be presented after some examples.

Example 3.3.8   Use the Cofactor Expansion Theorem along (a) row 1, (b) column 3 to find

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & 1 \\ 6 & 3 & 0 \end{vmatrix}.$$

Solution:

(a) We have

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & 1 \\ 6 & 3 & 0 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 6 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & -1 \\ 6 & 3 \end{vmatrix} = -6 + 18 + 36 = 48.$$

(b) We have

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & 1 \\ 6 & 3 & 0 \end{vmatrix} = 4\begin{vmatrix} 1 & -1 \\ 6 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ 6 & 3 \end{vmatrix} + 0 = 36 + 12 + 0 = 48. \qquad \square$$

Notice that (b) was easier than (a) in the previous example, because of the zero in column 3. Whenever one uses the cofactor expansion method to evaluate a determinant, it is usually best to select a row or column containing as many zeros as possible in order to minimize the amount of computation required.

Example 3.3.9 Evaluate

$$\begin{vmatrix} 0 & 3 & -1 & 0 \\ 5 & 0 & 8 & 2 \\ 7 & 2 & 5 & 4 \\ 6 & 1 & 7 & 0 \end{vmatrix}.$$

Solution: In this case, it is easiest to use either row 1 or column 4. Choosing row 1, we have

$$\begin{vmatrix} 0 & 3 & -1 & 0 \\ 5 & 0 & 8 & 2 \\ 7 & 2 & 5 & 4 \\ 6 & 1 & 7 & 0 \end{vmatrix} = -3\begin{vmatrix} 5 & 8 & 2 \\ 7 & 5 & 4 \\ 6 & 7 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 5 & 0 & 2 \\ 7 & 2 & 4 \\ 6 & 1 & 0 \end{vmatrix}$$
$$= -3\left[2(49 - 30) - 4(35 - 48) + 0\right] - \left[5(0 - 4) - 0 + 2(7 - 12)\right]$$
$$= -240.$$

In evaluating the determinants of order 3 on the right side of the first equality, we have used cofactor expansion along column 3 and row 1, respectively. For additional practice, the reader may wish to verify our result here by cofactor expansion along a different row or column. $\qquad \square$

Now we turn to the

Proof of the Cofactor Expansion Theorem: It follows from the definition of the determinant that $\det(A)$ can be written in the form

$$\det(A) = a_{i1}\hat{C}_{i1} + a_{i2}\hat{C}_{i2} + \cdots + a_{in}\hat{C}_{in} \qquad (3.3.1)$$

where the coefficients $\hat{C}_{ij}$ contain no elements from row i or column j. We must show that

$$\hat{C}_{ij} = C_{ij}$$

where C_{ij} is the cofactor of a_{ij}.

Consider first a_{11}. From Definition 3.1.8, the terms of $\det(A)$ that contain a_{11} are given by

$$a_{11}\sum \sigma(1, p_2, p_3, \ldots, p_n)a_{2p_2}a_{3p_3}\cdots a_{np_n},$$

where the summation is over the $(n-1)!$ distinct permutations of $2, 3, \ldots, n$. Thus,

$$\hat{C}_{11} = \sum \sigma(1, p_2, p_3, \ldots, p_n) a_{2p_2} a_{3p_3} \cdots a_{np_n}.$$

However, this summation is just the minor M_{11}, and since $C_{11} = M_{11}$, we have shown the coefficient of a_{11} in $\det(A)$ is indeed the cofactor C_{11}.

Now consider the element a_{ij}. By successively interchanging adjacent rows and columns of A, we can move a_{ij} into the $(1, 1)$ position *without altering the relative positions of the other rows and columns of* A. We let A' denote the resulting matrix. Obtaining A' from A requires $i - 1$ row interchanges and $j - 1$ column interchanges. Therefore, the total number of interchanges required to obtain A' from A is $i + j - 2$. Consequently,

$$\det(A) = (-1)^{i+j-2} \det(A') = (-1)^{i+j} \det(A').$$

Now for the key point. The coefficient of a_{ij} in $\det(A)$ must be $(-1)^{i+j}$ times the coefficient of a_{ij} in $\det(A')$. But, a_{ij} occurs in the $(1, 1)$ position of A', and so, as we have previously shown, its coefficient in $\det(A')$ is M'_{11}. Since the relative positions of the remaining rows in A have not altered, it follows that $M'_{11} = M_{ij}$, and therefore the coefficient of a_{ij} in $\det(A')$ is M_{ij}. Consequently, the coefficient of a_{ij} in $\det(A)$ is $(-1)^{i+j} M_{ij} = C_{ij}$. Applying this result to the elements $a_{i1}, a_{i2}, \ldots, a_{in}$ and comparing with (3.3.1) yields

$$\hat{C}_{ij} = C_{ij}, \qquad j = 1, 2, \ldots, n,$$

which establishes the theorem for expansion along a row. The result for expansion along a column follows directly, since $\det(A^T) = \det(A)$. ∎

We now have two computational methods for evaluating determinants: the use of elementary row operations given in the previous section to reduce the matrix in question to upper triangular form, and the Cofactor Expansion Theorem. In evaluating a given determinant by hand, it is usually most efficient (and least error prone) to use a combination of the two techniques. More specifically, we use elementary row operations to set all except one element in a row or column equal to zero and then use the Cofactor Expansion Theorem on that row or column. We illustrate with an example.

Example 3.3.10   Evaluate

$$\begin{vmatrix} 2 & 1 & 8 & 6 \\ 1 & 4 & 1 & 3 \\ -1 & 2 & 1 & 4 \\ 1 & 3 & -1 & 2 \end{vmatrix}.$$

Solution: We have

$$\begin{vmatrix} 2 & 1 & 8 & 6 \\ 1 & 4 & 1 & 3 \\ -1 & 2 & 1 & 4 \\ 1 & 3 & -1 & 2 \end{vmatrix} \overset{1}{=} \begin{vmatrix} 0 & -7 & 6 & 0 \\ 1 & 4 & 1 & 3 \\ 0 & 6 & 2 & 7 \\ 0 & -1 & -2 & -1 \end{vmatrix} \overset{2}{=} -\begin{vmatrix} -7 & 6 & 0 \\ 6 & 2 & 7 \\ -1 & -2 & -1 \end{vmatrix} \overset{3}{=} -\begin{vmatrix} -7 & 6 & 0 \\ -1 & -12 & 0 \\ -1 & -2 & -1 \end{vmatrix} \overset{4}{=} 90.$$

1. $A_{21}(-2)$, $A_{23}(1)$, $A_{24}(-1)$	2. Cofactor expansion along column 1
3. $A_{32}(7)$	4. Cofactor expansion along column 3

□

Example 3.3.11   Determine all values of k for which the system

$$
\begin{aligned}
10x_1 + kx_2 - x_3 &= 0,\\
kx_1 + x_2 - x_3 &= 0,\\
2x_1 + x_2 - 3x_3 &= 0,
\end{aligned}
$$

has nontrivial solutions.

Solution: We will apply Corollary 3.2.5. The determinant of the matrix of coefficients of the system is

$$
\det(A) = \begin{vmatrix} 10 & k & -1 \\ k & 1 & -1 \\ 2 & 1 & -3 \end{vmatrix} \overset{1}{=} \begin{vmatrix} 10 & k & -1 \\ k-10 & 1-k & 0 \\ -28 & 1-3k & 0 \end{vmatrix} \overset{2}{=} -\begin{vmatrix} k-10 & 1-k \\ -28 & 1-3k \end{vmatrix}
$$

$$
= -[(k-10)(1-3k) - (-28)(1-k)] = 3k^2 - 3k - 18 = 3(k^2 - k - 6)
$$

$$
= 3(k-3)(k+2).
$$

> **1.** $A_{12}(-1)$, $A_{13}(-3)$ **2.** Cofactor expansion along column 3.

From Corollary 3.2.5, the system has nontrivial solutions if and only if $\det(A) = 0$; that is, if and only if $k = 3$ or $k = -2$.   □

The Adjoint Method for A^{-1}

We next establish two corollaries to the Cofactor Expansion Theorem that, in the case of an invertible matrix A, lead to a method for expressing the elements of A^{-1} in terms of determinants.

Corollary 3.3.12   If the elements in the ith row (or column) of an $n \times n$ matrix A are multiplied by the cofactors of a different row (or column), then the sum of the resulting products is zero. That is,

1. If we use the elements of row i and the cofactors of row j,

$$
\sum_{k=1}^{n} a_{ik} C_{jk} = 0, \qquad i \neq j. \tag{3.3.2}
$$

2. If we use the elements of column i and the cofactors of column j,

$$
\sum_{k=1}^{n} a_{ki} C_{kj} = 0, \qquad i \neq j. \tag{3.3.3}
$$

Proof We prove (3.3.2). Let B be the matrix obtained from A by adding row i to row j $(i \neq j)$ in the matrix A. By P3, $\det(B) = \det(A)$. Cofactor expansion of B along row j gives

$$
\det(A) = \det(B) = \sum_{k=1}^{n} (a_{jk} + a_{ik}) C_{jk} = \sum_{k=1}^{n} a_{jk} C_{jk} + \sum_{k=1}^{n} a_{ik} C_{jk}.
$$

That is,

$$\det(A) = \det(A) + \sum_{k=1}^{n} a_{ik} C_{jk},$$

since by the Cofactor Expansion Theorem the first summation on the right-hand side is simply $\det(A)$. It follows immediately that

$$\sum_{k=1}^{n} a_{ik} C_{jk} = 0, \qquad i \neq j.$$

Equation (3.3.3) can be proved similarly (Problem 47). ∎

The Cofactor Expansion Theorem and the above corollary can be combined into the following corollary.

Corollary 3.3.13   Let A be an $n \times n$ matrix. If δ_{ij} is the Kronecker delta symbol (see Definition 2.2.19), then

$$\sum_{k=1}^{n} a_{ik} C_{jk} = \delta_{ij} \ \det(A), \qquad \sum_{k=1}^{n} a_{ki} C_{kj} = \delta_{ij} \ \det(A). \qquad (3.3.4)$$

The formulas in (3.3.4) should be reminiscent of the index form of the matrix product. Combining this with the fact that the Kronecker delta gives the elements of the identity matrix, we might suspect that (3.3.4) is telling us something about the inverse of A. Before establishing that this suspicion is indeed correct, we need a definition.

DEFINITION   3.3.14

If every element in an $n \times n$ matrix A is replaced by its cofactor, the resulting matrix is called the **matrix of cofactors** and is denoted M_C. The transpose of the matrix of cofactors, M_C^T, is called the **adjoint** of A and is denoted adj(A). Thus, the elements of adj(A) are

$$\mathrm{adj}(A)_{ij} = C_{ji}.$$

Example 3.3.15   Determine adj(A) if

$$A = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 5 & 4 \\ 3 & -2 & 0 \end{bmatrix}.$$

Solution: We first determine the cofactors of A:

$$C_{11} = 8, \quad C_{12} = 12, \quad C_{13} = -13, \quad C_{21} = 6, \quad C_{22} = 9, \quad C_{23} = 4,$$
$$C_{31} = 15, \quad C_{32} = -5, \quad C_{33} = 10.$$

Thus,

$$M_C = \begin{bmatrix} 8 & 12 & -13 \\ 6 & 9 & 4 \\ 15 & -5 & 10 \end{bmatrix},$$

so that

$$\text{adj}(A) = M_C^T = \begin{bmatrix} 8 & 6 & 15 \\ 12 & 9 & -5 \\ -13 & 4 & 10 \end{bmatrix}. \qquad \square$$

We can now prove the next theorem.

Theorem 3.3.16   **(The Adjoint Method for Computing A^{-1})**

If $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

Proof Let $B = \dfrac{1}{\det(A)} \text{adj}(A)$. Then we must establish that $AB = I_n = BA$. But, using the index form of the matrix product,

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} a_{ik} \cdot \frac{1}{\det(A)} \cdot \text{adj}(A)_{kj} = \frac{1}{\det(A)} \sum_{k=1}^{n} a_{ik} C_{jk} = \delta_{ij},$$

where we have used Equation (3.3.4) in the last step. Consequently, $AB = I_n$. We leave it as an exercise (Problem 53) to verify that $BA = I_n$ also. ∎

Example 3.3.17   For the matrix in Example 3.3.15,

$$\det(A) = 55,$$

so that

$$A^{-1} = \frac{1}{55} \begin{bmatrix} 8 & 6 & 15 \\ 12 & 9 & -5 \\ -13 & 4 & 10 \end{bmatrix}. \qquad \square$$

For square matrices of relatively small size, the adjoint method for computing A^{-1} is often easier than using elementary row operations to reduce A to upper triangular form.

In Chapter 7, we will find that the solution of a system of *differential* equations can be expressed naturally in terms of matrix functions. Certain problems will require us to find the inverse of such matrix functions. For 2×2 systems, the adjoint method is very quick.

Example 3.3.18   Find A^{-1} if $A = \begin{bmatrix} e^{2t} & e^{-t} \\ 3e^{2t} & 6e^{-t} \end{bmatrix}$.

Solution: In this case,

$$\det(A) = (e^{2t})(6e^{-t}) - (3e^{2t})(e^{-t}) = 3e^{t},$$

and

$$\text{adj}(A) = \begin{bmatrix} 6e^{-t} & -e^{-t} \\ -3e^{2t} & e^{2t} \end{bmatrix},$$

so that

$$A^{-1} = \begin{bmatrix} 2e^{-2t} & -\frac{1}{3}e^{-2t} \\ -e^{t} & \frac{1}{3}e^{t} \end{bmatrix}. \qquad \square$$

Cramer's Rule

We now derive a technique that enables us, in the case when $\det(A) \neq 0$, to express the unique solution of an $n \times n$ linear system

$$A\mathbf{x} = \mathbf{b}$$

directly in terms of determinants. Let B_k denote the matrix obtained by replacing the kth column vector of A with $\mathbf{b}$. Thus,

$$B_k = \begin{bmatrix} a_{11} & a_{12} & \ldots & b_1 & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & b_2 & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & b_n & \ldots & a_{nn} \end{bmatrix}.$$

The key point to notice is that the cofactors of the elements in the kth column of B_k coincide with the corresponding cofactors of A. Thus, expanding $\det(B_k)$ along the kth column using the Cofactor Expansion Theorem yields

$$\det(B_k) = b_1 C_{1k} + b_2 C_{2k} + \cdots + b_n C_{nk} = \sum_{i=1}^{n} b_i C_{ik}, \quad k = 1, 2, \ldots, n, \quad (3.3.5)$$

where the C_{ij} are the cofactors of A. We can now prove Cramer's rule.

Theorem 3.3.19   **(Cramer's Rule)**

If $\det(A) \neq 0$, the unique solution to the $n \times n$ system $A\mathbf{x} = \mathbf{b}$ is $(x_1, x_2, \ldots, x_n)$, where

$$x_k = \frac{\det(B_k)}{\det(A)}, \quad k = 1, 2, \ldots, n. \quad (3.3.6)$$

Proof If $\det(A) \neq 0$, then the system $A\mathbf{x} = \mathbf{b}$ has the unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}, \quad (3.3.7)$$

where, from Theorem 3.3.16, we can write

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A). \quad (3.3.8)$$

If we let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

and recall that $\operatorname{adj}(A)_{ij} = C_{ji}$, then substitution from (3.3.8) into (3.3.7) and use of the index form of the matrix product yields

$$x_k = \sum_{i=1}^{n} (A^{-1})_{ki} b_i = \sum_{i=1}^{n} \frac{1}{\det(A)} \operatorname{adj}(A)_{ki} b_i$$

$$= \frac{1}{\det(A)} \sum_{i=1}^{n} C_{ik} b_i, \qquad k = 1, 2, \ldots, n.$$

Using (3.3.5), we can write this as

$$x_k = \frac{\det(B_k)}{\det(A)}, \qquad k = 1, 2, \ldots, n$$

as required.   ■

Remark   In general, Cramer's rule requires more work than the Gaussian elimination method, and it is restricted to $n \times n$ systems whose coefficient matrix is invertible. However, it is a powerful theoretical tool, since it gives us a formula for the solution of an $n \times n$ system, provided $\det(A) \neq 0$.

Example 3.3.20   Solve

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 4, \\ x_1 & + & x_2 & - & 5x_3 & = & -3, \\ -2x_1 & - & x_2 & + & 4x_3 & = & 0. \end{array}$$

Solution: The following determinants are easily evaluated:

$$\det(A) = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 1 & -5 \\ -2 & -1 & 4 \end{vmatrix} = 8, \qquad \det(B_1) = \begin{vmatrix} 4 & 2 & -1 \\ -3 & 1 & -5 \\ 0 & -1 & 4 \end{vmatrix} = 17,$$

$$\det(B_2) = \begin{vmatrix} 3 & 4 & -1 \\ 1 & -3 & -5 \\ -2 & 0 & 4 \end{vmatrix} = -6, \qquad \det(B_3) = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 1 & -3 \\ -2 & -1 & 0 \end{vmatrix} = 7.$$

Inserting these results into (3.3.6) yields $x_1 = \frac{17}{8}$, $x_2 = -\frac{6}{8} = -\frac{3}{4}$, and $x_3 = \frac{7}{8}$, so that the solution to the system is $(\frac{17}{8}, -\frac{3}{4}, \frac{7}{8})$.   □

Exercises for 3.3

Key Terms

Minor, Cofactor, Cofactor expansion, Matrix of cofactors, Adjoint, Cramer's rule.

Skills

- Be able to compute the minors and cofactors of a matrix.

- Understand the difference between M_{ij} and C_{ij}.

- Be able to compute the determinant of a matrix via cofactor expansion.

- Be able to compute the matrix of cofactors and the adjoint of a matrix.

- Be able to use the adjoint of an invertible matrix A to compute A^{-1}.

- Be able to use Cramer's rule to solve a linear system of equations.

True-False Review

For Questions 1–7, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. The $(2, 3)$-minor of a matrix is the same as the $(2, 3)$-cofactor of the matrix.

2. We have $A \cdot \mathrm{adj}(A) = \det(A) \cdot I_n$ for all $n \times n$ matrices A.

3. Cofactor expansion of a matrix along any row or column will yield the same result, although the individual terms in the expansion along different rows or columns can vary.

4. If A is an $n \times n$ matrix and c is a scalar, then

$$\text{adj}(cA) = c \cdot \text{adj}(A).$$

5. If A and B are 2×2 matrices, then

$$\text{adj}(A + B) = \text{adj}(A) + \text{adj}(B).$$

6. If A and B are 2×2 matrices, then

$$\text{adj}(AB) = \text{adj}(A) \cdot \text{adj}(B).$$

7. For every n, $\text{adj}(I_n) = I_n$.

Problems

For Problems 1–3, determine all minors and cofactors of the given matrix.

1. $A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$.

2. $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 4 \\ 2 & 1 & 5 \end{bmatrix}$.

3. $A = \begin{bmatrix} 2 & 10 & 3 \\ 0 & -1 & 0 \\ 4 & 1 & 5 \end{bmatrix}$.

4. If

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 3 & 4 & 1 & 2 \\ 7 & 1 & 4 & 6 \\ 5 & 0 & 1 & 2 \end{bmatrix},$$

determine the minors M_{12}, M_{31}, M_{23}, M_{42}, and the corresponding cofactors.

For Problems 5–10, use the Cofactor Expansion Theorem to evaluate the given determinant along the specified row or column.

5. $\begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix}$, row 1.

6. $\begin{vmatrix} -1 & 2 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 4 \end{vmatrix}$, column 3.

7. $\begin{vmatrix} 2 & 1 & -4 \\ 7 & 1 & 3 \\ 1 & 5 & -2 \end{vmatrix}$, row 2.

8. $\begin{vmatrix} 3 & 1 & 4 \\ 7 & 1 & 2 \\ 2 & 3 & -5 \end{vmatrix}$, column 1.

9. $\begin{vmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{vmatrix}$, row 3.

10. $\begin{vmatrix} 1 & -2 & 3 & 0 \\ 4 & 0 & 7 & -2 \\ 0 & 1 & 3 & 4 \\ 1 & 5 & -2 & 0 \end{vmatrix}$, column 4.

For Problems 11–19, evaluate the given determinant using the techniques of this section.

11. $\begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & -1 \\ 7 & 2 & 5 \end{vmatrix}$.

12. $\begin{vmatrix} -1 & 2 & 3 \\ 0 & 1 & 4 \\ 2 & -1 & 3 \end{vmatrix}$.

13. $\begin{vmatrix} 2 & -1 & 3 \\ 5 & 2 & 1 \\ 3 & -3 & 7 \end{vmatrix}$.

14. $\begin{vmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{vmatrix}$.

15. $\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix}$.

16. $\begin{vmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 3 \\ 0 & 2 & -1 & 0 \\ 1 & 3 & -2 & 4 \end{vmatrix}$.

17. $\begin{vmatrix} 3 & 5 & 2 & 6 \\ 2 & 3 & 5 & -5 \\ 7 & 5 & -3 & -16 \\ 9 & -6 & 27 & -12 \end{vmatrix}$.

18. $\begin{vmatrix} 2 & -7 & 4 & 3 \\ 5 & 5 & -3 & 7 \\ 6 & 2 & 6 & 3 \\ 4 & 2 & -4 & 5 \end{vmatrix}$.

19. $\begin{vmatrix} 2 & 0 & -1 & 3 & 0 \\ 0 & 3 & 0 & 1 & 2 \\ 0 & 1 & 3 & 0 & 4 \\ 1 & 0 & 1 & -1 & 0 \\ 3 & 0 & 2 & 0 & 5 \end{vmatrix}$.

20. If

$$A = \begin{bmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ -z & 1 & -1 & 0 \end{bmatrix},$$

show that $\det(A) = (x + y + z)^2$.

21. (a) Consider the 3×3 *Vandermonde* determinant $V(r_1, r_2, r_3)$ defined by

$$V(r_1, r_2, r_3) = \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix}.$$

Show that

$$V(r_1, r_2, r_3) = (r_2 - r_1)(r_3 - r_1)(r_3 - r_2).$$

(b) More generally, show that the $n \times n$ Vandermonde determinant

$$V(r_1, r_2, \ldots, r_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & \vdots & & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix}$$

has value

$$V(r_1, r_2, \ldots, r_n) = \prod_{1 \le i < m \le n} (r_m - r_i).$$

For Problems 22–31, find **(a)** $\det(A)$, **(b)** the matrix of cofactors M_C, **(c)** adj(A), and, if possible, **(d)** A^{-1}.

22. $A = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$.

23. $A = \begin{bmatrix} -1 & -2 \\ 4 & 1 \end{bmatrix}$.

24. $A = \begin{bmatrix} 5 & 2 \\ -15 & -6 \end{bmatrix}$.

25. $A = \begin{bmatrix} 2 & -3 & 0 \\ 2 & 1 & 5 \\ 0 & -1 & 2 \end{bmatrix}$.

26. $A = \begin{bmatrix} -2 & 3 & -1 \\ 2 & 1 & 5 \\ 0 & 2 & 3 \end{bmatrix}$.

27. $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 4 \\ 5 & 1 & 7 \end{bmatrix}$.

28. $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$.

29. $A = \begin{bmatrix} 2 & -3 & 5 \\ 1 & 2 & 1 \\ 0 & 7 & -1 \end{bmatrix}$.

30. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$.

31. $A = \begin{bmatrix} 1 & 0 & 3 & 5 \\ -2 & 1 & 1 & 3 \\ 3 & 9 & 0 & 2 \\ 2 & 0 & 3 & -1 \end{bmatrix}$.

32. Let $A = \begin{bmatrix} 1 & -2x & 2x^2 \\ 2x & 1 - 2x^2 & -2x \\ 2x^2 & 2x & 1 \end{bmatrix}$.

(a) Show that $\det(A) = (1 + 2x^2)^3$.

(b) Use the adjoint method to find A^{-1}.

In Problems 33–35, find the specified element in the *inverse* of the given matrix. Do not use elementary row operations.

33. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$; $(3, 2)$-element.

34. $A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{bmatrix}$; $(3, 1)$-element.

35. $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ -1 & 1 & 2 & 0 \end{bmatrix}$; $(2, 3)$-element.

In Problems 36–38, find A^{-1}.

36. $A = \begin{bmatrix} 3e^t & e^{2t} \\ 2e^t & 2e^{2t} \end{bmatrix}$.

37. $A = \begin{bmatrix} e^t \sin 2t & -e^{-t} \cos 2t \\ e^t \cos 2t & e^{-t} \sin 2t \end{bmatrix}$.

38. $A = \begin{bmatrix} e^t & te^t & e^{-2t} \\ e^t & 2te^t & e^{-2t} \\ e^t & te^t & 2e^{-2t} \end{bmatrix}$.

39. If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix},$$

compute the matrix product $A \cdot \text{adj}(A)$. What can you conclude about $\det(A)$?

For Problems 40–43, use Cramer's rule to solve the given linear system.

40. $\begin{aligned} 2x_1 - 3x_2 &= 2, \\ x_1 + 2x_2 &= 4. \end{aligned}$

41. $\begin{aligned} 3x_1 - 2x_2 + x_3 &= 4, \\ x_1 + x_2 - x_3 &= 2, \\ x_1 \quad\quad + x_3 &= 1. \end{aligned}$

42. $\begin{aligned} x_1 - 3x_2 + x_3 &= 0, \\ x_1 + 4x_2 - x_3 &= 0, \\ 2x_1 + x_2 - 3x_3 &= 0. \end{aligned}$

43. $\begin{aligned} x_1 - 2x_2 + 3x_3 - x_4 &= 1, \\ 2x_1 \quad\quad + x_3 \quad\quad &= 2, \\ x_1 + x_2 \quad\quad - x_4 &= 0, \\ x_2 - 2x_3 + x_4 &= 3. \end{aligned}$

44. Use Cramer's rule to determine x_1 and x_2 if

$$\begin{aligned} e^t x_1 + e^{-2t} x_2 &= 3\sin t, \\ e^t x_1 - 2e^{-2t} x_2 &= 4\cos t. \end{aligned}$$

45. Determine the value of x_2 such that

$$\begin{aligned} x_1 + 4x_2 - 2x_3 + x_4 &= 2, \\ 2x_1 + 9x_2 - 3x_3 - 2x_4 &= 5, \\ x_1 + 5x_2 + x_3 - x_4 &= 3, \\ 3x_1 + 14x_2 + 7x_3 - 2x_4 &= 6. \end{aligned}$$

46. Find all solutions to the system

$$\begin{aligned} (b + c)x_1 + a(x_2 + x_3) &= a, \\ (c + a)x_1 + b(x_3 + x_1) &= b, \\ (a + b)x_1 + c(x_1 + x_2) &= c, \end{aligned}$$

where a, b, c are constants. Make sure you consider all cases (that is, those when there is a unique solution, an infinite number of solutions, and no solutions).

47. Prove Equation (3.3.3).

48. ◇ Let A be a randomly generated invertible 4×4 matrix. Verify the Cofactor Expansion Theorem for expansion along row 1.

49. ◇ Let A be a randomly generated 4×4 matrix. Verify Equation (3.3.3) when $i = 2$ and $j = 4$.

50. ◇ Let A be a randomly generated 5×5 matrix. Determine $\text{adj}(A)$ and compute $A \cdot \text{adj}(A)$. Use your result to determine $\det(A)$.

51. ◇ Solve the system of equations

$$\begin{aligned} 1.21x_1 + 3.42x_2 + 2.15x_3 &= 3.25, \\ 5.41x_1 + 2.32x_2 + 7.15x_3 &= 4.61, \\ 21.63x_1 + 3.51x_2 + 9.22x_3 &= 9.93. \end{aligned}$$

Round answers to two decimal places.

52. ◇ Use Cramer's rule to solve the system $A\mathbf{x} = \mathbf{b}$ if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ 4 & 4 & 3 & 2 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 68 \\ -72 \\ -87 \\ 79 \\ 43 \end{bmatrix}.$$

53. Verify that $BA = I_n$ in the proof of Theorem 3.3.16.

3.4 Summary of Determinants

The primary aim of this section is to serve as a stand-alone introduction to determinants for readers who desire only a cursory review of the major facts pertaining to determinants. It may also be used as a review of the results derived in Sections 3.1–3.3.

Formulas for the Determinant

The determinant of an $n \times n$ matrix A, denoted $\det(A)$, is a scalar whose value can be obtained in the following manner.

1. If $A = [a_{11}]$, then $\det(A) = a_{11}$.