14. On $\mathbb{R}^2$, define the operation of addition by

$$(x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2).$$

Do axioms A5 and A6 in the definition of a vector space hold? Justify your answer.

15. On $M_2(\mathbb{R})$, define the operation of addition by

$$A + B = AB,$$

and use the usual scalar multiplication operation. Determine which axioms for a vector space are satisfied by $M_2(\mathbb{R})$ with the above operations.

16. On $M_2(\mathbb{R})$, define the operations of addition and multiplication by a real number ($\oplus$ and $\cdot$, respectively) as follows:

$$A \oplus B = -(A + B),$$
$$k \cdot A = -kA,$$

where the operations on the right-hand sides of these equations are the usual ones associated with $M_2(\mathbb{R})$.

Determine which of the axioms for a vector space are satisfied by $M_2(\mathbb{R})$ with the operations $\oplus$ and $\cdot$.

For Problems 17–18, verify that the given set of objects together with the usual operations of addition and scalar multiplication is a *complex* vector space.

17. $\mathbb{C}^2$.

18. $M_2(\mathbb{C})$, the set of all 2×2 matrices with complex entries.

19. Is $\mathbb{C}^3$ a *real* vector space? Explain.

20. Is $\mathbb{R}^3$ a *complex* vector space? Explain.

21. Prove part 3 of Theorem 4.2.6.

22. Prove part 6 of Theorem 4.2.6.

23. Prove that P_n is a vector space.

4.3 Subspaces

Let us try to make contact between the abstract vector space idea and the solution of an applied problem. Vector spaces generally arise as the sets containing the unknowns in a given problem. For example, if we are solving a differential equation, then the basic unknown is a function, and therefore any solution to the differential equation will be an element of the vector space V of all functions defined on an appropriate interval. Consequently, the solution set of a differential equation is a subset of V. Similarly, consider the system of linear equations $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix with real elements. The basic unknown in this system, $\mathbf{x}$, is a column n-vector, or equivalently a vector in $\mathbb{R}^n$. Consequently, the solution set to the system is a subset of the vector space $\mathbb{R}^n$. As these examples illustrate, the solution set of an applied problem is generally a subset of vectors from an appropriate vector space (schematically represented in Figure 4.3.1). The question we will need to answer in the future is whether this subset of vectors is a vector space in its own right. The following definition introduces the terminology we will use:

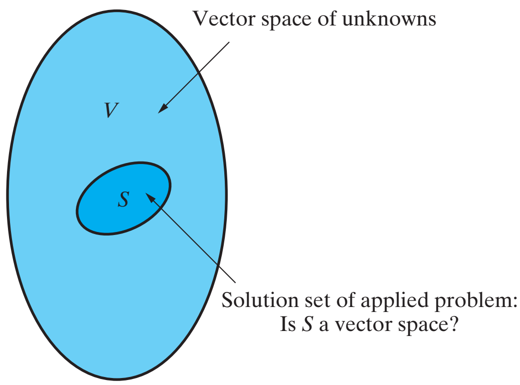

Figure 4.3.1: The solution set S of an applied problem is a subset of the vector space V of unknowns in the problem.

DEFINITION 4.3.1

Let S be a nonempty subset of a vector space V. If S is itself a vector space under the same operations of addition and scalar multiplication as used in V, then we say that S is a **subspace** of V.

In establishing that a given subset S of vectors from a vector space V is a subspace of V, it would appear as though we must check that each axiom in the vector space definition is satisfied when we restrict our attention to vectors lying only in S. The first and most important theorem of the section tells us that all we need do, in fact, is check the closure axioms A1 and A2. If these are satisfied, then the remaining axioms necessarily hold in S. This is a very useful theorem that will be applied on several occasions throughout the remainder of the text.

Theorem 4.3.2 Let S be a nonempty subset of a vector space V. Then S is a subspace of V if and only if S is closed under the operations of addition and scalar multiplication in V.

Proof If S is a subspace of V, then it is a vector space, and hence, it is certainly closed under addition and scalar multiplication. Conversely, assume that S is closed under addition and scalar multiplication. We must prove that Axioms A3–A10 of Definition 4.2.1 hold when we restrict to vectors in S. Consider first the axioms A3, A4, and A7–A10. These are properties of the addition and scalar multiplication operations, hence since we use the same operations in S as in V, these axioms are all inherited from V by the subset S. Finally, we establish A5 and A6: Choose any vector[1] $\mathbf{u}$ in S. Since S is closed under scalar multiplication, both $0\mathbf{u}$ and $(-1)\mathbf{u}$ are in S. But by Theorem 4.2.6, $0\mathbf{u} = \mathbf{0}$ and $(-1)\mathbf{u} = -\mathbf{u}$, hence $\mathbf{0}$ and $-\mathbf{u}$ are both in S. Therefore, A5 and A6 are satisfied. ■

The idea behind Theorem 4.3.2 is that once we have a vector space V in place, then any nonempty subset S, equipped with the same addition and scalar multiplication operations, will inherit all of the axioms that involve those operations. The only possible concern we have for S is whether or not it satisfies the closure axioms A1 and A2. Of course, we presumably had to carry out the full verification of A1–A10 for the vector space V in the first place, before gaining the shortcut of Theorem 4.3.2 for the subset S.

In determining whether a subset S of a vector space V is a subspace of V, we must keep clear in our minds what the given vector space is and what conditions on the vectors in V restrict them to lie in the subset S. This is most easily done by expressing S in set notation as follows:

$$S = \{\mathbf{v} \in V : \text{ conditions on } \mathbf{v}\}.$$

We illustrate with an example.

Example 4.3.3 Verify that the set of all real solutions to the following linear system is a subspace of $\mathbb{R}^3$:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0, \\ 2x_1 + 5x_2 - 4x_3 &= 0. \end{aligned}$$

Solution: The reduced row-echelon form of the augmented matrix of the system is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix},$$

[1] This is possible since S is assumed to be nonempty.

so that the solution set of the system is

$$S = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = (-3r, 2r, r),\ \ r \in \mathbb{R}\},$$

which is a nonempty subset of $\mathbb{R}^3$. We now use Theorem 4.3.2 to verify that S is a subspace of $\mathbb{R}^3$: If $\mathbf{x} = (-3r, 2r, r)$ and $\mathbf{y} = (-3s, 2s, s)$ are any two vectors in S, then

$$\mathbf{x} + \mathbf{y} = (-3r, 2r, r) + (-3s, 2s, s) = (-3(r+s), 2(r+s), r+s) = (-3t, 2t, t),$$

where $t = r+s$. Thus, $\mathbf{x}+\mathbf{y}$ meets the required form for elements of S, and consequently, if we add two vectors in S, the result is another vector in S. Similarly, if we multiply an arbitrary vector $\mathbf{x} = (-3r, 2r, r)$ in S by a real number k, the resulting vector is

$$k\mathbf{x} = k(-3r, 2r, r) = (-3kr, 2kr, kr) = (-3w, 2w, w),$$

where $w = kr$. Hence, $k\mathbf{x}$ again has the proper form for membership in the subset S, and so S is closed under scalar multiplication. By Theorem 4.3.2, S is a subspace of $\mathbb{R}^3$. Note, of course, that our application of Theorem 4.3.2 hinges on our prior knowledge that $\mathbb{R}^3$ is a vector space.

Geometrically, the vectors in S lie along the line of intersection of the planes with the given equations. This is the line through the origin in the direction of the vector $\mathbf{v} = (-3, 2, 1)$. (See Figure 4.3.2.)

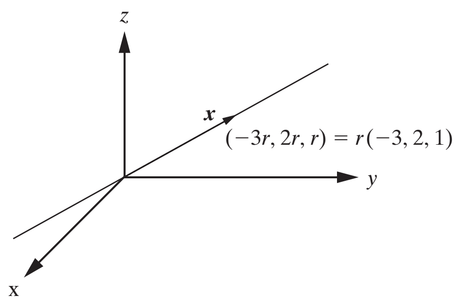

Figure 4.3.2: The solution set to the homogeneous system of linear equations in Example 4.3.3 is a subspace of $\mathbb{R}^3$.

□

Example 4.3.4 Verify that $S = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = (r, -3r+1),\ \ r \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^2$.

Solution: One approach here, according to Theorem 4.3.2, is to demonstrate the failure of closure under addition or scalar multiplication. For example, if we start with two vectors in S, say $\mathbf{x} = (r, -3r+1)$ and $\mathbf{y} = (s, -3s+1)$, then

$$\mathbf{x} + \mathbf{y} = (r, -3r+1) + (s, -3s+1) = (r+s, -3(r+s)+2) = (w, -3w+2),$$

where $w = r+s$. We see that $\mathbf{x}+\mathbf{y}$ does not have the required form for membership in S. Hence, S is not closed under addition and therefore fails to be a subspace of $\mathbb{R}^2$. Alternatively, we can show similarly that S is not closed under scalar multiplication.

Observant readers may have noticed another reason that S cannot form a subspace. Geometrically, the points in S correspond to those points that lie on the line with Cartesian equation $y = -3x+1$. Since this line does not pass through the origin, S does not contain the zero vector $\mathbf{0} = (0, 0)$, and therefore we know S cannot be a subspace. □

Remark In general, we have the following important observation.

> If a subset S of a vector space V fails to contain the zero vector $\mathbf{0}$, then it cannot form a subspace.

This observation can often be made more quickly than deciding whether or not S is closed under addition and closed under scalar multiplication. However, we caution that if the zero vector *does* belong to S, then the observation is inconclusive and further investigation is required to determine whether or not S forms a subspace of V.

Example 4.3.5 Let S denote the set of all real symmetric $n \times n$ matrices. Verify that S is a subspace of $M_n(\mathbb{R})$.

Solution: The subset of interest is

$$S = \{A \in M_n(\mathbb{R}) : A^T = A\}.$$

Note that S is nonempty, since, for example, it contains the zero matrix 0_n. We now verify closure of S under addition and scalar multiplication. Let A and B be in S. Then

$$A^T = A \qquad \text{and} \qquad B^T = B.$$

Using these conditions and the properties of the transpose yields

$$(A + B)^T = A^T + B^T = A + B$$

and

$$(kA)^T = kA^T = kA$$

for all real values of k. Consequently $A + B$ and kA are both symmetric matrices, so they are elements of S. Hence S is closed under both addition and scalar multiplication and so is indeed a subspace of $M_n(\mathbb{R})$. □

Remark Notice in Example 4.3.5 that it was not necessary to actually write out the matrices A and B in terms of their elements $[a_{ij}]$ and $[b_{ij}]$, respectively. This shows the advantage of using simple abstract notation to describe the elements of the subset S in some situations.

Example 4.3.6 Let V be the vector space of all real-valued functions defined on an interval $[a, b]$, and let S denote the set of all functions in V that satisfy $f(a) = 0$. Verify that S is a subspace of V.

Solution: We have

$$S = \{f \in V : f(a) = 0\},$$

which is nonempty since it contains, for example, the zero function

$$O(x) = 0 \qquad \text{for all } x \text{ in } [a, b].$$

Assume that f and g are in S, so that $f(a) = 0$ and $g(a) = 0$. We now check for closure of S under addition and scalar multiplication. We have

$$(f + g)(a) = f(a) + g(a) = 0 + 0 = 0,$$

which implies that $f + g \in S$. Hence, S is closed under addition. Further, if k is any real number,

$$(kf)(a) = kf(a) = k0 = 0,$$

so that S is also closed under scalar multiplication. Theorem 4.3.2 therefore implies that S is a subspace of V. Some representative functions from S are sketched in Figure 4.3.3. □

In the next theorem, we establish that the subset $\{\mathbf{0}\}$ of a vector space V is in fact a subspace of V. We call this subspace the **trivial subspace** of V.

Theorem 4.3.7 Let V be a vector space with zero vector $\mathbf{0}$. Then $S = \{\mathbf{0}\}$ is a subspace of V.

Proof Note that S is nonempty. Further, the closure of S under addition and scalar multiplication follow, respectively, from

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \qquad \text{and} \qquad k\mathbf{0} = \mathbf{0},$$

where the second statement follows from Theorem 4.2.6. ■

We now use Theorem 4.3.2 to establish an important result pertaining to homogeneous systems of linear equations that has already been illustrated in Example 4.3.3.

Theorem 4.3.8 Let A be an $m \times n$ matrix. The solution set of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{C}^n$.

Proof Let S denote the solution set of the homogeneous linear system. Then we can write

$$S = \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{0}\},$$

a subset of $\mathbb{C}^n$. Since a homogeneous system always admits the trivial solution $\mathbf{x} = \mathbf{0}$, we know that S is nonempty. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are in S, then

$$A\mathbf{x}_1 = \mathbf{0} \qquad \text{and} \qquad A\mathbf{x}_2 = \mathbf{0}.$$

Using properties of the matrix product, we have

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

so that $\mathbf{x}_1 + \mathbf{x}_2$ also solves the system and therefore is in S. Furthermore, if k is any complex scalar, then

$$A(k\mathbf{x}) = kA\mathbf{x} = k\mathbf{0} = \mathbf{0},$$

so that $k\mathbf{x}$ is also a solution of the system and therefore is in S. Since S is closed under both addition and scalar multiplication, it follows from Theorem 4.3.2 that S is a subspace of $\mathbb{C}^n$. ■

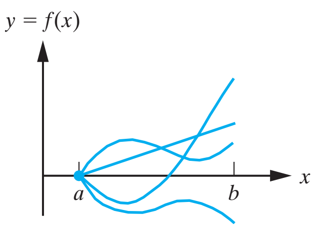

Figure 4.3.3: Representative functions in the subspace S given in Example 4.3.6. Each function in S satisfies $f(a) = 0$.

The preceding theorem has established that the solution set to any homogeneous linear system of equations is a vector space. Owing to the importance of this vector space, it is given a special name.

DEFINITION 4.3.9

Let A be an $m \times n$ matrix. The solution set to the corresponding homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is called the **null space of** A and is denoted nullspace(A). Thus,

$$\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{0}\}.$$

Remarks

1. If the matrix A has real elements, then we will consider only the corresponding real solutions to $A\mathbf{x} = \mathbf{0}$. Consequently, in this case,

$$\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\},$$

a subspace of $\mathbb{R}^n$.

2. The previous theorem *does not* hold for the solution set of a *nonhomogeneous* linear system $A\mathbf{x} = \mathbf{b}$, for $\mathbf{b} \neq \mathbf{0}$, since $\mathbf{x} = \mathbf{0}$ is not in the solution set of the system.

Next we introduce the vector space of primary importance in the study of linear differential equations. This vector space arises as a subspace of the vector space of all functions that are defined on an interval I.

Example 4.3.10 Let V denote the vector space of all functions that are defined on an interval I, and let $C^k(I)$ denote the set of all functions that are continuous and have (at least) k continuous derivatives on the interval I, for a fixed non-negative integer k. Show that $C^k(I)$ is a subspace of V.

Solution: In this case

$$C^k(I) = \{f \in V : f, f', f'', \dots, f^{(k)} \text{ exist and are continuous on } I\}.$$

This set is nonempty, as the zero function $O(x) = 0$ for all $x \in I$ is an element of $C^k(I)$. Moreover, it follows from the properties of derivatives that if we add two functions in $C^k(I)$, the result is a function in $C^k(I)$. Similarly, if we multiply a function in $C^k(I)$ by a scalar, then the result is a function in $C^k(I)$. Thus, Theorem 4.3.2 implies that $C^k(I)$ is a subspace of V. □

Our final result in this section ties together the ideas introduced here with the theory of differential equations.

Theorem 4.3.11 The set of all solutions to the homogeneous linear differential equation

$$y'' + a_1(x)y' + a_2(x)y = 0 \qquad (4.3.1)$$

on an interval I is a vector space.

Proof Let S denote the set of all solutions to the given differential equation. Then S is a nonempty subset of $C^2(I)$, since the identically zero function $y = 0$ is a solution to the differential equation. We establish that S is in fact a subspace of[2] $C^k(I)$. Let y_1 and y_2 be in S, and let k be a scalar. Then we have the following:

$$y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0 \quad \text{and} \quad y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0. \qquad (4.3.2)$$

Now, if $y(x) = y_1(x) + y_2(x)$, then

$$\begin{aligned} y'' + a_1y' + a_2y &= (y_1 + y_2)'' + a_1(x)(y_1 + y_2)' + a_2(x)(y_1 + y_2) \\ &= [y_1'' + a_1(x)y_1' + a_2(x)y_1] + [y_2'' + a_1(x)y_2' + a_2(x)y_2] \\ &= 0 + 0 = 0, \end{aligned}$$

[2]It is important at this point that we have already established Example 4.3.10, so that S is a subset of a set that is indeed a vector space.

where we have used (4.3.2). Consequently, $y(x) = y_1(x) + y_2(x)$ is a solution to the differential equation (4.3.1). Moreover, if $y(x) = ky_1(x)$, then

$$\begin{aligned} y'' + a_1 y' + a_2 y &= (ky_1)'' + a_1(x)(ky_1)' + a_2(x)(ky_1) \\ &= k[y_1'' + a_1(x)y_1' + a_2(x)y_1] = 0, \end{aligned}$$

where we have once more used (4.3.2). This establishes that $y(x) = ky_1(x)$ is a solution to Equation (4.3.1). Therefore, S is closed under both addition and scalar multiplication. Consequently, the set of all solutions to Equation (4.3.1) is a subspace of $C^2(I)$. ■

We will refer to the set of all solutions to a differential equation of the form (4.3.1) as the **solution space** of the differential equation. A key theoretical result that we will establish in Chapter 6 regarding the homogeneous linear differential equation (4.3.1) is that *every* solution to the differential equation has the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where y_1, y_2 are any two nonproportional solutions. The power of this result is impressive: It reduces the search for all solutions to Equation (4.3.1) to the search for just two nonproportional solutions. In vector space terms, the result can be restated as follows:

> Every vector in the solution space to the differential equation (4.3.1) can be written as a linear combination of any two nonproportional solutions y_1 and y_2.

We say that the solution space is **spanned** by y_1 and y_2. Moreover, two nonproportional solutions are referred to as **linearly independent**. For example, we saw in Example 1.2.16 that the set of all solutions to the differential equation

$$y'' + \omega^2 y = 0$$

is spanned by $y_1(x) = \cos \omega x$, and $y_2(x) = \sin \omega x$, and y_1 and y_2 are linearly independent. We now begin our investigation as to whether this type of idea will work more generally when the solution set to a problem is a vector space. For example, what about the solution set to a homogeneous linear system $A\mathbf{x} = \mathbf{0}$? We might suspect that if there are k free variables defining the vectors in nullspace(A), then every solution to $A\mathbf{x} = \mathbf{0}$ can be expressed as a linear combination of k basic solutions. We will establish that this is indeed the case in Section 4.9. The two key concepts we need to generalize are (1) spanning a general vector space with a set of vectors, and (2) linear independence in a general vector space. These will be addressed in turn in the next two sections.

Exercises for 4.3

Key Terms

Subspace, Trivial subspace, Null space of a matrix A.

Skills

- Be able to check whether or not a subset S of a vector space V is a subspace of V.
- Be able to compute the null space of an $m \times n$ matrix A.

True-False Review

For Questions 1–8, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. The null space of an $m \times n$ matrix A with real elements is a subspace of $\mathbb{R}^m$.

2. The solution set of any linear system of m equations in n variables forms a subspace of $\mathbb{C}^n$.

3. The points in $\mathbb{R}^2$ that lie on the line $y = mx + b$ form a subspace of $\mathbb{R}^2$ if and only if $b = 0$.

4. If $m < n$, then $\mathbb{R}^m$ is a subspace of $\mathbb{R}^n$.

5. A nonempty set S of a vector space V that is closed under scalar multiplication contains the zero vector of V.

6. If $V = \mathbb{R}$ is a vector space under the usual operations of addition and scalar multiplication, then the subset $\mathbb{R}^+$ of *positive* real numbers, together with the operations defined in Problem 12 of Section 4.2, forms a subspace of V.

7. If $V = \mathbb{R}^3$ and S consists of all points on the xy-plane, the xz-plane, and the yz-plane, then S is a subspace of V.

8. If V is a vector space, then two different subspaces of V can contain no common vectors other than $\mathbf{0}$.

Problems

1. Let $S = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = (2k, -3k), k \in \mathbb{R}\}$.

(a) Establish that S is a subspace of $\mathbb{R}^2$.

(b) Make a sketch depicting the subspace S in the Cartesian plane.

2. Let $S = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = (r - 2s, 3r + s, s), r, s \in \mathbb{R}\}$.

(a) Establish that S is a subspace of $\mathbb{R}^3$.

(b) Show that the vectors in S lie on the plane with equation $3x - y + 7z = 0$.

For Problems 3–19, express S in set notation and determine whether it is a subspace of the given vector space V.

3. $V = \mathbb{R}^2$, and S is the set of all vectors (x, y) in V satisfying $3x + 2y = 0$.

4. $V = \mathbb{R}^4$, and S is the set of all vectors of the form $(x_1, 0, x_3, 2)$.

5. $V = \mathbb{R}^3$, and S is the set of all vectors (x, y, z) in V satisfying $x + y + z = 1$.

6. $V = \mathbb{R}^n$, and S is the set of all solutions to the nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$, where A is a fixed $m \times n$ matrix and $\mathbf{b}$ ($\neq \mathbf{0}$) is a fixed vector.

7. $V = \mathbb{R}^2$, and S consists of all vectors (x, y) satisfying $x^2 - y^2 = 0$.

8. $V = M_2(\mathbb{R})$, and S is the subset of all 2×2 matrices with $\det(A) = 1$.

9. $V = M_n(\mathbb{R})$, and S is the subset of all $n \times n$ lower triangular matrices.

10. $V = M_n(\mathbb{R})$, and S is the subset of all $n \times n$ invertible matrices.

11. $V = M_2(\mathbb{R})$, and S is the subset of all 2×2 symmetric matrices.

12. $V = M_2(\mathbb{R})$, and S is the subset of all 2×2 skew-symmetric matrices.

13. V is the vector space of all real-valued functions defined on the interval $[a, b]$, and S is the subset of V consisting of all functions satisfying $f(a) = f(b)$.

14. V is the vector space of all real-valued functions defined on the interval $[a, b]$, and S is the subset of V consisting of all functions satisfying $f(a) = 1$.

15. V is the vector space of all real-valued functions defined on the interval $(-\infty, \infty)$, and S is the subset of V consisting of all functions satisfying $f(-x) = f(x)$ for all $x \in (-\infty, \infty)$.

16. $V = P_2$, and S is the subset of P_2 consisting of all polynomials of the form $p(x) = ax^2 + b$.

17. $V = P_2$, and S is the subset of P_2 consisting of all polynomials of the form $p(x) = ax^2 + 1$.

18. $V = C^2(I)$, and S is the subset of V consisting of those functions satisfying the differential equation

$$y'' + 2y' - y = 0$$

on I.

19. $V = C^2(I)$, and S is the subset of V consisting of those functions satisfying the differential equation

$$y'' + 2y' - y = 1$$

on I.

For Problems 20–22, determine the null space of the given matrix A.

20. $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & -2 \\ -1 & 3 & 4 \end{bmatrix}$.

21. $A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 3 & 10 & -4 & 6 \\ 2 & 5 & -6 & -1 \end{bmatrix}$.

22. $A = \begin{bmatrix} 1 & i & -2 \\ 3 & 4i & -5 \\ -1 & -3i & i \end{bmatrix}$.

23. Show that the set of all solutions to the nonhomogeneous differential equation

$$y'' + a_1 y' + a_2 y = F(x),$$

where $F(x)$ is nonzero on an interval I, is not a subspace of $C^2(I)$.

24. Let S_1 and S_2 be subspaces of a vector space V. Let

$$S_1 \cup S_2 = \{\mathbf{v} \in V : \mathbf{v} \in S_1 \text{ or } \mathbf{v} \in S_2\},$$
$$S_1 \cap S_2 = \{\mathbf{v} \in V : \mathbf{v} \in S_1 \text{ and } \mathbf{v} \in S_2\},$$

and let

$$S_1 + S_2 = \{\mathbf{v} \in V : \mathbf{v} = \mathbf{x} + \mathbf{y} \text{ for some } \mathbf{x} \in S_1 \text{ and } \mathbf{y} \in S_2\}.$$

(a) Show that, in general, $S_1 \cup S_2$ is not a subspace of V.

(b) Show that $S_1 \cap S_2$ is a subspace of V.

(c) Show that $S_1 + S_2$ is a subspace of V.

4.4 Spanning Sets

The only algebraic operations that are defined in a vector space V are those of addition and scalar multiplication. Consequently, the most general way in which we can combine the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ in V is

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k, \tag{4.4.1}$$

where $c_1, c_2, \dots, c_k$ are scalars. An expression of the form (4.4.1) is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. Since V is closed under addition and scalar multiplication, it follows that the foregoing linear combination is itself a vector in V. One of the questions we wish to answer is whether every vector in a vector space can be obtained by taking linear combinations of a finite set of vectors. The following terminology is used in the case when the answer to this question is affirmative:

DEFINITION 4.4.1

If *every* vector in a vector space V can be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2, \dots, \mathbf{v}_k$, we say that V is **spanned** or **generated** by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ and call the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ a **spanning set** for V. In this case, we also say that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ **spans** V.

This spanning idea was introduced in the preceding section within the framework of differential equations. In addition, we are all used to representing geometric vectors in $\mathbb{R}^3$ in terms of their components as (see Section 4.1)

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ denote the unit vectors pointing along the positive x-, y-, and z-axes, respectively, of a rectangular Cartesian coordinate system. Using the above terminology, we say that $\mathbf{v}$ has been expressed as a linear combination of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, and that the vector space of all geometric vectors is spanned by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

We now consider several examples to illustrate the spanning concept in different vector spaces.

Example 4.4.2 Show that $\mathbb{R}^2$ is spanned by the vectors

$$\mathbf{v}_1 = (1, 1) \quad \text{and} \quad \mathbf{v}_2 = (2, -1).$$