32. $\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1, \mathbf{v}_2$ are collinear vectors in $\mathbb{R}^3$.

33. Prove that if S and S' are subsets of a vector space V such that S is a subset of S', then $\text{span}(S)$ is a subset of $\text{span}(S')$.

34. Prove that

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

if and only if $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

4.5 Linear Dependence and Linear Independence

As indicated in the previous section, in analyzing a vector space we will be interested in determining a spanning set. The reader has perhaps already noticed that a vector space V can have many such spanning sets.

Example 4.5.1 Observe that $\{(1, 0), (0, 1)\}$, $\{(1, 0), (1, 1)\}$, and $\{(1, 0), (0, 1), (1, 2)\}$ are all spanning sets for $\mathbb{R}^2$. □

As another illustration, two different spanning sets for $V = M_2(\mathbb{R})$ were given in Example 4.4.5 and the remark that followed. Given the abundance of spanning sets available for a given vector space V, we are faced with a natural question: Is there a "best class of" spanning sets to use? The answer, to a large degree, is "yes". For instance, in Example 4.5.1, the spanning set $\{(1, 0), (0, 1), (1, 2)\}$ contains an "extra" vector, $(1, 2)$, which seems to be unnecessary for spanning $\mathbb{R}^2$, since $\{(1, 0), (0, 1)\}$ is already a spanning set. In some sense, $\{(1, 0), (0, 1)\}$ is a more efficient spanning set. It is what we call a ***minimal* spanning set**, since it contains the minimum number of vectors needed to span the vector space.[3]

But how will we know if we have found a minimal spanning set (assuming one exists)? Returning to the example above, we have seen that

$$\text{span}\{(1, 0), (0, 1)\} = \text{span}\{(1, 0), (0, 1), (1, 2)\} = \mathbb{R}^2.$$

Observe that the vector $(1, 2)$ is already a linear combination of $(1, 0)$ and $(0, 1)$, and therefore it does not add any new vectors to the linear span of $\{(1, 0), (0, 1)\}$.

As a second example, consider the vectors $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (3, -2, 1)$, and $\mathbf{v}_3 = 4\mathbf{v}_1 + \mathbf{v}_2 = (7, 2, 5)$. It is easily verified that $\det([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]) = 0$. Consequently, the three vectors lie in a plane (see Figure 4.5.1) and therefore, since they are not collinear, the linear span of these three vectors is the whole of this plane. Furthermore, the same plane is generated if we consider the linear span of $\mathbf{v}_1$ and $\mathbf{v}_2$ alone. As in the previous example, the reason that $\mathbf{v}_3$ does not add any new vectors to the linear span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ is that it is already a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. It is not possible, however, to generate all vectors in the plane by taking linear combinations of just one vector, as we could generate only a line lying in the plane in that case. Consequently, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a minimal spanning set for the subspace of $\mathbb{R}^3$ consisting of all points lying on the plane.

As a final example, recall from Example 1.2.16 that the solution space to the differential equation

$$y'' + y = 0$$

[3] Since a single (nonzero) vector in $\mathbb{R}^2$ spans only the line through the origin along which it points, it cannot span all of $\mathbb{R}^2$; hence, the minimum number of vectors required to span $\mathbb{R}^2$ is 2.

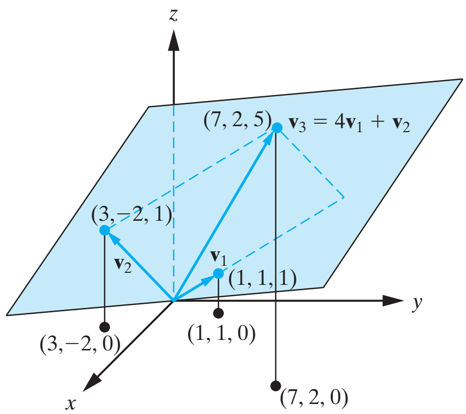

Figure 4.5.1: $\mathbf{v}_3 = 4\mathbf{v}_1 + \mathbf{v}_2$ lies in the plane through the origin containing $\mathbf{v}_1$ and $\mathbf{v}_2$, and so, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

can be written as $\text{span}\{y_1, y_2\}$, where $y_1(x) = \cos x$ and $y_2(x) = \sin x$. However, if we let $y_3(x) = 3\cos x - 2\sin x$, for instance, then $\{y_1, y_2, y_3\}$ is also a spanning set for the solution space of the differential equation, since

$$\begin{aligned}\text{span}\{y_1, y_2, y_3\} &= \{c_1 \cos x + c_2 \sin x + c_3(3\cos x - 2\sin x) : \; c_1, c_2, c_3 \in \mathbb{R}\}\\ &= \{(c_1 + 3c_3)\cos x + (c_2 - 2c_3)\sin x : \; c_1, c_2, c_3 \in \mathbb{R}\}\\ &= \{d_1 \cos x + d_2 \sin x : \; d_1, d_2 \in \mathbb{R}\}\\ &= \text{span}\{y_1, y_2\}.\end{aligned}$$

The reason that $\{y_1, y_2, y_3\}$ is not a *minimal* spanning set for the solution space is that y_3 is a linear combination of y_1 and y_2, and therefore, as we have just shown, it does not add any new vectors to the linear span of $\{\cos x, \sin x\}$.

More generally, it is not too difficult to extend the argument used in the preceding examples to establish the following general result.

Theorem 4.5.2 Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of at least two vectors in a vector space V. If one of the vectors in the set is a linear combination of the other vectors in the set, then that vector can be deleted from the given set of vectors and the linear span of the resulting set of vectors will be the same as the linear span of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$.

Proof The proof of this result is left for the exercises (Problem 48). ■

For instance, if $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_k$, then Theorem 4.5.2 says that

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} = \text{span}\{\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_k\}.$$

In this case, the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is not a minimal spanning set.

To determine a minimal spanning set, the problem we face in view of Theorem 4.5.2 is that of determining when a vector in $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ can be expressed as a linear combination of the remaining vectors in the set. The correct formulation for solving this problem requires the concepts of linear dependence and linear independence, which we are now ready to introduce. First we consider linear dependence.

DEFINITION 4.5.3

A finite nonempty set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in a vector space V is said to be **linearly dependent** if there exist scalars $c_1, c_2, \dots, c_k$, *not all zero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Such a nontrivial linear combination of vectors is sometimes referred to as a **linear dependency** among the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$.

A set of vectors that is not linearly dependent is called linearly independent. This can be stated mathematically as follows:

DEFINITION 4.5.4

A finite, nonempty set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in a vector space V is said to be **linearly independent** if the *only* values of the scalars $c_1, c_2, \dots, c_k$ for which

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

are $c_1 = c_2 = \cdots = c_k = 0$.

Remarks

1. It follows immediately from the preceding two definitions that a nonempty set of vectors in a vector space V is linearly independent if and only if it is not linearly dependent.
2. If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a linearly independent set of vectors, we sometimes informally say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are themselves linearly independent. The same remark applies to the linearly dependent condition as well.

Consider the simple case of a set containing a single vector $\mathbf{v}$. If $\mathbf{v} = \mathbf{0}$, then $\{\mathbf{v}\}$ is linearly dependent, since for any nonzero scalar c_1,

$$c_1\mathbf{0} = \mathbf{0}.$$

On the other hand, if $\mathbf{v} \neq \mathbf{0}$, then the only value of the scalar c_1 for which

$$c_1\mathbf{v} = \mathbf{0}$$

is $c_1 = 0$. Consequently, $\{\mathbf{v}\}$ is linearly independent. We can therefore state the next theorem.

Theorem 4.5.5 A set consisting of a single vector $\mathbf{v}$ in a vector space V is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$. Therefore, any set consisting of a single *nonzero* vector is linearly independent.

We next establish that linear dependence of a set containing at least two vectors is equivalent to the property that we are interested in—namely, that at least one vector in the set can be expressed as a linear combination of the remaining vectors in the set.

Theorem 4.5.6 Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of at least two vectors in a vector space V. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent if and only if at least one of the vectors in the set can be expressed as a linear combination of the others.

Proof If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent, then according to Definition 4.5.3, there exist scalars $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Suppose that $c_i \neq 0$. Then we can express $\mathbf{v}_i$ as a linear combination of the other vectors as follows:

$$\mathbf{v}_i = -\frac{1}{c_i}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{i-1}\mathbf{v}_{i-1} + c_{i+1}\mathbf{v}_{i+1} + \cdots + c_k\mathbf{v}_k).$$

Conversely, suppose that one of the vectors, say, $\mathbf{v}_j$, can be expressed as a linear combination of the remaining vectors. That is,

$$\mathbf{v}_j = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{j-1}\mathbf{v}_{j-1} + c_{j+1}\mathbf{v}_{j+1} + \cdots + c_k\mathbf{v}_k.$$

Adding $(-1)\mathbf{v}_j$ to both sides of this equation yields

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{j-1}\mathbf{v}_{j-1} - \mathbf{v}_j + c_{j+1}\mathbf{v}_{j+1} + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Since the coefficient of $\mathbf{v}_j$ is $-1 \neq 0$, the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent. ■

As far as the minimal-spanning-set idea is concerned, Theorems 4.5.6 and 4.5.2 tell us that a linearly dependent spanning set for a (nontrivial) vector space V cannot be a minimal spanning set. On the other hand, we will see in the next section that a linearly independent spanning set for V *must* be a minimal spanning set for V. For the remainder of this section, however, we focus more on the mechanics of determining whether a given set of vectors is linearly independent or linearly dependent. Sometimes this can be done by inspection. For example, Figure 4.5.2 illustrates that any set of three vectors in $\mathbb{R}^2$ is linearly dependent.

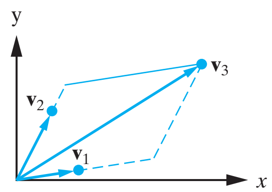

Figure 4.5.2: The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent in $\mathbb{R}^2$, since $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

As another example, let V be the vector space of all functions defined on an interval I. If

$$f_1(x) = 1, \qquad f_2(x) = 2\sin^2 x, \qquad f_3(x) = -5\cos^2 x,$$

then $\{f_1, f_2, f_3\}$ is linearly dependent in V, since the identity $\sin^2 x + \cos^2 x = 1$ implies that for all $x \in I$,

$$f_1(x) = \tfrac{1}{2}f_2(x) - \tfrac{1}{5}f_3(x).$$

We can therefore conclude from Theorem 4.5.2 that

$$\text{span}\{1, 2\sin^2 x, -5\cos^2 x\} = \text{span}\{2\sin^2 x, -5\cos^2 x\}.$$

In relatively simple examples, the following general results can be applied. They are a direct consequence of the definition of linearly dependent vectors and are left for the exercises (Problem 49).

Proposition 4.5.7 Let V be a vector space.

1. Any set of *two* vectors in V is linearly dependent if and only if the vectors are proportional.

2. Any set of vectors in V containing the zero vector is linearly dependent.

Remark We emphasize that the first result in Proposition 4.5.7 holds only for the case of two vectors. It cannot be applied to sets containing more than two vectors.

Example 4.5.8 If $\mathbf{v}_1 = (1, 2, -9)$ and $\mathbf{v}_2 = (-2, -4, 18)$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent in $\mathbb{R}^3$, since $\mathbf{v}_2 = -2\mathbf{v}_1$. Geometrically, $\mathbf{v}_1$ and $\mathbf{v}_2$ lie on the same line. □

Example 4.5.9 If

$$A_1 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 2 & 5 \\ -3 & 2 \end{bmatrix},$$

then $\{A_1, A_2, A_3\}$ is linearly dependent in $M_2(\mathbb{R})$, since it contains the zero vector from $M_2(\mathbb{R})$. □

For more complicated situations, we must resort to Definitions 4.5.3 and 4.5.4, although conceptually it is always helpful to keep in mind that the essence of the problem we are solving is to determine whether a vector in a given set can be expressed as a linear combination of the remaining vectors in the set. We now give some examples to illustrate the use of Definitions 4.5.3 and 4.5.4.

Example 4.5.10 If $\mathbf{v}_1 = (1, 2, -1)$ $\mathbf{v}_2 = (2, -1, 1)$, and $\mathbf{v}_3 = (8, 1, 1)$, show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent in $\mathbb{R}^3$, and determine the linear dependency relationship.

Solution: We must first establish that there are values of the scalars c_1, c_2, c_3, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}. \tag{4.5.1}$$

Substituting for the given vectors yields

$$c_1(1, 2, -1) + c_2(2, -1, 1) + c_3(8, 1, 1) = (0, 0, 0).$$

That is,

$$(c_1 + 2c_2 + 8c_3, 2c_1 - c_2 + c_3, -c_1 + c_2 + c_3) = (0, 0, 0).$$

Equating corresponding components on either side of this equation yields

$$\begin{aligned} c_1 + 2c_2 + 8c_3 &= 0, \\ 2c_1 - c_2 + c_3 &= 0, \\ -c_1 + c_2 + c_3 &= 0. \end{aligned}$$

The reduced row-echelon form of the augmented matrix of this system is

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consequently, the system has an infinite number of solutions for c_1, c_2, c_3, so the vectors are linearly dependent.

In order to determine a specific linear dependency relationship, we proceed to find c_1, c_2, and c_3. Setting $c_3 = t$, we have $c_2 = -3t$ and $c_1 = -2t$. Taking $t = 1$ and substituting these values for c_1, c_2, c_3 into (4.5.1), we obtain the linear dependency relationship

$$-2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0},$$

or equivalently,

$$\mathbf{v}_1 = -\tfrac{3}{2}\mathbf{v}_2 + \tfrac{1}{2}\mathbf{v}_3,$$

which can be easily verified using the given expressions for $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. It follows from Theorem 4.5.2 that

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_2, \mathbf{v}_3\}.$$

Geometrically, we can conclude that $\mathbf{v}_1$ lies in the plane determined by the vectors $\mathbf{v}_2$ and $\mathbf{v}_3$. □

Example 4.5.11 Determine whether the following matrices are linearly dependent or linearly independent in $M_2(\mathbb{R})$:

$$A_1 = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

Solution: The condition for determining whether these vectors are linearly dependent or linearly independent,

$$c_1A_1 + c_2A_2 + c_3A_3 = 0_2,$$

is equivalent in this case to

$$c_1\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} + c_2\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + c_3\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which is satisfied if and only if

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0, \\ -c_1 + c_2 - c_3 &= 0, \\ 2c_1 \qquad + 2c_3 &= 0, \\ 3c_2 + c_3 &= 0. \end{aligned}$$

The reduced row-echelon form of the augmented matrix of this homogeneous system is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which implies that the system has only the trivial solution $c_1 = c_2 = c_3 = 0$. It follows from Definition 4.5.4 that $\{A_1, A_2, A_3\}$ is linearly independent. □

As a corollary to Theorem 4.5.2, we establish the following result.

Corollary 4.5.12 Any nontrivial, finite set of linearly dependent vectors in a vector space V contains a linearly independent subset that has the same linear span as the given set of vectors.

Proof Since the given set is linearly dependent, at least one of the vectors in the set is a linear combination of the remaining vectors, by Theorem 4.5.6. Thus, by Theorem 4.5.2, we can delete that vector from the set, and the resulting set of vectors will span the same subspace of V as the original set. If the resulting set is linearly independent, then we are done. If not, then we can repeat the procedure to eliminate another vector in the set. Continuing in this manner (with a finite number of iterations), we will obtain a linearly independent set that spans the same subspace of V as the subspace spanned by the original set of vectors. ■

Remark Corollary 4.5.12 is actually true even if the set of vectors in question is infinite, but we shall not need to consider that case in this text. In the case of an infinite set of vectors, other techniques are required for the proof.

Note that the linearly independent set obtained using the procedure given in the previous theorem is not unique, and therefore the question arises whether the number of vectors in any resulting linearly independent set is independent of the manner in which the procedure is applied. We will give an affirmative answer to this question in Section 4.6.

Example 4.5.13 Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-1, 1, 4)$, $\mathbf{v}_3 = (3, 3, 2)$, and $\mathbf{v}_4 = (-2, -4, -6)$. Determine a linearly independent set of vectors that spans the same subspace of $\mathbb{R}^3$ as $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Solution: Setting

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

requires that

$$c_1(1, 2, 3) + c_2(-1, 1, 4) + c_3(3, 3, 2) + c_4(-2, -4, -6) = (0, 0, 0),$$

leading to the linear system

$$\begin{aligned} c_1 - c_2 + 3c_3 - 2c_4 &= 0, \\ 2c_1 + c_2 + 3c_3 - 4c_4 &= 0, \\ 3c_1 + 4c_2 + 2c_3 - 6c_4 &= 0. \end{aligned}$$

The augmented matrix of this system is

$$\begin{bmatrix} 1 & -1 & 3 & -2 & 0 \\ 2 & 1 & 3 & -4 & 0 \\ 3 & 4 & 2 & -6 & 0 \end{bmatrix}$$

and the reduced row-echelon form of the augmented matrix of this system is

$$\begin{bmatrix} 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system has two free variables, $c_3 = s$ and $c_4 = t$, and so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent. Then $c_2 = s$ and $c_1 = 2t - 2s$. So the general form of the solution is

$$(2t - 2s, s, s, t) = s(-2, 1, 1, 0) + t(2, 0, 0, 1).$$

Setting $s = 1$ and $t = 0$ yields the linear combination

$$-2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}, \tag{4.5.2}$$

and setting $s = 0$ and $t = 1$ yields the linear combination

$$2\mathbf{v}_1 + \mathbf{v}_4 = \mathbf{0}. \tag{4.5.3}$$

We can solve (4.5.2) for $\mathbf{v}_3$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$, and we can solve (4.5.3) for $\mathbf{v}_4$ in terms of $\mathbf{v}_1$. Hence, according to Theorem 4.5.2, we have

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

By Proposition 4.5.7, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the linearly independent set we are seeking. Geometrically, the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is a plane, and the vectors $\mathbf{v}_3$ and $\mathbf{v}_4$ lie in this plane. □

Linear Dependence and Linear Independence in $\mathbb{R}^n$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of vectors in $\mathbb{R}^n$, and let A denote the matrix that has $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ as *column* vectors. Thus,

$$A = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]. \tag{4.5.4}$$

Since each of the given vectors is in $\mathbb{R}^n$, it follows that A has n rows and is therefore an $n \times k$ matrix.

The linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ can be written in matrix form as (see Theorem 2.2.9)

$$A\mathbf{c} = \mathbf{0}, \tag{4.5.5}$$

where A is given in Equation (4.5.4) and $\mathbf{c} = [c_1\ c_2\ \dots\ c_k]^T$. Consequently, we can state the following theorem and corollary:

Theorem 4.5.14 Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$ and $A = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly dependent if and only if the linear system $A\mathbf{c} = \mathbf{0}$ has a nontrivial solution.

Corollary 4.5.15 Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$ and $A = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$.

1. If $k > n$, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly dependent.
2. If $k = n$, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly dependent if and only if $\det(A) = 0$.

Proof If $k > n$, the system (4.5.5) has an infinite number of solutions (see Corollary 2.5.11), hence the vectors are linearly dependent by Theorem 4.5.14.

On the other hand, if $k = n$, the system (4.5.5) is $n \times n$, and hence, from Corollary 3.2.5, it has an infinite number of solutions if and only if $\det(A) = 0$. ■

Example 4.5.16 Determine whether the given vectors are linearly dependent or linearly independent in $\mathbb{R}^4$.

1. $\mathbf{v}_1 = (1, 3, -1, 0)$, $\mathbf{v}_2 = (2, 9, -1, 3)$, $\mathbf{v}_3 = (4, 5, 6, 11)$, $\mathbf{v}_4 = (1, -1, 2, 5)$, $\mathbf{v}_5 = (3, -2, 6, 7)$.
2. $\mathbf{v}_1 = (1, 4, 1, 7)$, $\mathbf{v}_2 = (3, -5, 2, 3)$, $\mathbf{v}_3 = (2, -1, 6, 9)$, $\mathbf{v}_4 = (-2, 3, 1, 6)$.

Solution:

1. Since we have five vectors in $\mathbb{R}^4$, Corollary 4.5.15 implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is necessarily linearly dependent.
2. In this case, we have four vectors in $\mathbb{R}^4$, and therefore, we can use the determinant:

$$\det(A) = \det[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4] = \begin{vmatrix} 1 & 3 & 2 & -2 \\ 4 & -5 & -1 & 3 \\ 1 & 2 & 6 & 1 \\ 7 & 3 & 9 & 6 \end{vmatrix} = -462.$$

Since the determinant is nonzero, it follows from Corollary 4.5.15 that the given set of vectors is linearly independent. □

Linear Independence of Functions

We now consider the general problem of determining whether or not a given set of functions is linearly independent or linearly dependent. We begin by specializing the general Definition 4.5.4 to the case of a set of functions defined on an interval I.

DEFINITION 4.5.17

The set of functions $\{f_1, f_2, \ldots, f_k\}$ is **linearly independent on an interval** I if and only if the only values of the scalars $c_1, c_2, \ldots, c_k$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) = 0, \qquad \text{for all } x \in I, \tag{4.5.6}$$

are $c_1 = c_2 = \cdots = c_k = 0$.

The main point to notice is that the condition (4.5.6) must hold for all x in I.

A key tool in deciding whether or not a collection of functions is linearly independent on an interval I is the Wronskian. As we will see in Chapter 6, we can draw particularly sharp conclusions from the Wronskian about the linear dependence or independence of a family of *solutions to a linear homogeneous differential equation*.

DEFINITION 4.5.18

Let $f_1, f_2, \ldots, f_k$ be functions in $C^{k-1}(I)$. The **Wronskian** of these functions is the order k determinant defined by

$$W[f_1, f_2, \ldots, f_k](x) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_k(x) \\ f_1'(x) & f_2'(x) & \ldots & f_k'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \ldots & f_k^{(k-1)}(x) \end{vmatrix}.$$

Remark Notice that the Wronskian is a function defined on I. Also note that this function depends on the order of the functions in the Wronskian. For example, using properties of determinants,

$$W[f_2, f_1, \ldots, f_k](x) = -W[f_1, f_2, \ldots, f_k](x).$$

Example 4.5.19 If $f_1(x) = \sin x$ and $f_2(x) = \cos x$ on $(-\infty, \infty)$, then

$$W[f_1, f_2](x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = (\sin x)(-\sin x) - (\cos x)(\cos x)$$
$$= -(\sin^2 x + \cos^2 x) = -1. \qquad \square$$

Example 4.5.20 If $f_1(x) = x$, $f_2(x) = x^2$, and $f_3(x) = x^3$ on $(-\infty, \infty)$, then

$$W[f_1, f_2, f_3](x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = x(12x^2 - 6x^2) - (6x^3 - 2x^3) = 2x^3. \qquad \square$$

We can now state and prove the main result about the Wronskian.

Theorem 4.5.21 Let $f_1, f_2, \ldots, f_k$ be functions in $C^{k-1}(I)$. If $W[f_1, f_2, \ldots, f_k]$ is nonzero at some point x_0 in I, then $\{f_1, f_2, \ldots, f_k\}$ is linearly independent on I.

Proof To apply Definition 4.5.17, assume that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) = 0,$$

for all x in I. Then, differentiating $k - 1$ times yields the linear system

$$\begin{array}{lllll} c_1 f_1(x) & + c_2 f_2(x) & + \cdots & + c_k f_k(x) & = 0, \\ c_1 f_1'(x) & + c_2 f_2'(x) & + \cdots & + c_k f_k'(x) & = 0, \\ & & & & \vdots \\ c_1 f_1^{(k-1)}(x) & + c_2 f_2^{(k-1)}(x) & + \cdots & + c_k f_k^{(k-1)}(x) & = 0, \end{array}$$

where the unknowns in the system are $c_1, c_2, \ldots, c_k$. We wish to show that $c_1 = c_2 = \cdots = c_k = 0$. The determinant of the matrix of coefficients of this system is just $W[f_1, f_2, \ldots, f_k](x)$. Consequently, if $W[f_1, f_2, \ldots, f_k](x_0) \neq 0$ for some x_0 in I, then the determinant of the matrix of coefficients of the system is nonzero at that point, and therefore the only solution to the system is the trivial solution $c_1 = c_2 = \cdots = c_k = 0$. That is, the given set of functions is linearly independent on I. ■

Remarks

1. Notice that it is only necessary for $W[f_1, f_2, \ldots, f_k](x)$ to be nonzero at one point in I for $\{f_1, f_2, \ldots, f_k\}$ to be linearly independent on I.

2. Theorem 4.5.21 *does not say* that if $W[f_1, f_2, \ldots, f_k](x) = 0$ for every x in I, then $\{f_1, f_2, \ldots, f_k\}$ is linearly dependent on I. As we will see in the next example below, the Wronskian of a linearly independent set of functions on an interval I can be identically zero on I. Instead, the logical equivalent of the preceding theorem is: If $\{f_1, f_2, \ldots, f_k\}$ is linearly dependent on I, then $W[f_1, f_2, \ldots, f_k](x) = 0$ at every point of I.

If $W[f_1, f_2, \ldots, f_k](x) = 0$ for all x in I, Theorem 4.5.21 gives no information as to the linear dependence or independence of $\{f_1, f_2, \ldots, f_k\}$ on I.

Example 4.5.22 Determine whether the following functions are linearly dependent or linearly independent on $I = (-\infty, \infty)$.

(a) $f_1(x) = e^x$, $f_2(x) = x^2e^x$.

(b) $f_1(x) = x$, $f_2(x) = x + x^2$, $f_3(x) = 2x - x^2$.

(c) $f_1(x) = x^2$, $f_2(x) = \begin{cases} 2x^2, & \text{if } x \geq 0, \\ -x^2, & \text{if } x < 0. \end{cases}$

Solution:

(a)

$$W[f_1, f_2](x) = \begin{vmatrix} e^x & x^2e^x \\ e^x & e^x(x^2+2x) \end{vmatrix} = e^{2x}(x^2+2x) - x^2e^{2x} = 2xe^{2x}.$$

Since $W[f_1, f_2](x) \neq 0$ (except at $x = 0$), the functions are linearly independent on $(-\infty, \infty)$.

(b)

$$\begin{aligned} W[f_1, f_2, f_3](x) &= \begin{vmatrix} x & x+x^2 & 2x-x^2 \\ 1 & 1+2x & 2-2x \\ 0 & 2 & -2 \end{vmatrix} \\ &= x\left[(-2)(1+2x) - 2(2-2x)\right] \\ &\quad - \left[(-2)(x+x^2) - 2(2x-x^2)\right] = 0. \end{aligned}$$

Thus, no conclusion can be drawn from Theorem 4.5.21. However, a closer inspection of the functions reveals, for example, that

$$f_2 = 3f_1 - f_3.$$

Consequently, the functions are linearly dependent on $(-\infty, \infty)$.

(c) If $x \geq 0$, then

$$W[f_1, f_2](x) = \begin{vmatrix} x^2 & 2x^2 \\ 2x & 4x \end{vmatrix} = 0,$$

whereas if $x < 0$, then

$$W[f_1, f_2](x) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0.$$

Thus, $W[f_1, f_2](x) = 0$ for all x in $(-\infty, \infty)$, so no conclusion can be drawn from Theorem 4.5.21. Again we take a closer look at the given functions. They are sketched in Figure 4.5.3. In this case, we see that on the interval $(-\infty, 0)$, the functions are linearly dependent, since

$$f_1 + f_2 = 0.$$

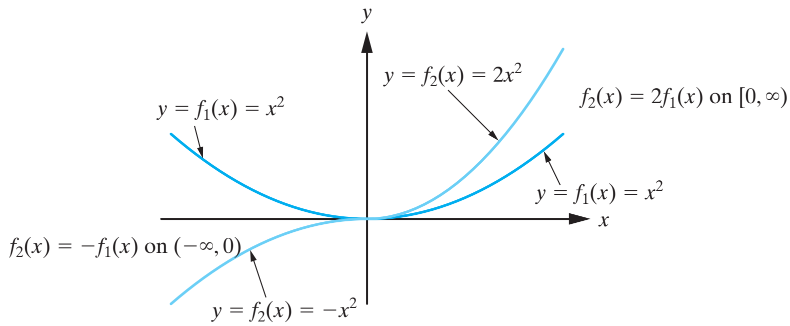

Figure 4.5.3: Two functions that are linearly independent on $(-\infty, \infty)$, but whose Wronskian is identically zero on that interval.

They are also linearly dependent on $[0, \infty)$, since on this interval we have

$$2f_1 - f_2 = 0.$$

The key point is to realize that there is no set of *nonzero* constants c_1, c_2 for which

$$c_1 f_1 + c_2 f_2 = 0$$

holds *for all* x in $(-\infty, \infty)$. Hence, the given functions are linearly independent on $(-\infty, \infty)$. This illustrates our second remark following Theorem 4.5.21, and it emphasizes the importance of the role played by the interval I when discussing linear dependence and linear independence of functions. A collection of functions may be linearly independent on an interval I_1, but linearly dependent on another interval I_2. □

It might appear at this stage that the usefulness of the Wronskian is questionable, since if $W[f_1, f_2, \ldots, f_k]$ vanishes on an interval I, then no conclusion can be drawn as to the linear dependence or linear independence of the functions $f_1, f_2, \ldots, f_k$ on I. However, the real power of the Wronskian is in its application to solutions of linear differential equations of the form

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0. \tag{4.5.7}$$

In Chapter 6, we will establish that if we have n functions that are *solutions of an equation of the form* (4.5.7) on an interval I, then if the Wronskian of these functions is identically zero on I, the functions are indeed linearly dependent on I. Thus, the Wronskian does completely characterize the linear dependence or linear independence of solutions of such equations. This is a fundamental result in the theory of linear differential equations.

Exercises for 4.5

Key Terms

Linearly dependent set, Linear dependency, Linearly independent set, Minimal spanning set, Wronskian of a set of functions.

Skills

- Be able to determine whether a given finite set of vectors is linearly dependent or linearly independent. For sets of one or two vectors, you should be able to do this *at a glance*. If the set is linearly dependent, be able to determine a linear dependency relationship among the vectors.
- Be able to take a linearly dependent set of vectors and remove vectors until it becomes a linearly independent set of vectors with the same span as the original set.

- Be able to produce a linearly independent set of vectors that spans a given subspace of a vector space V.
- Be able to conclude immediately that a set of k vectors in $\mathbb{R}^n$ is linearly dependent if $k > n$, and know what can be said in the case where $k = n$ as well.
- Know what information the Wronskian does (and does not) give about the linear dependence or linear independence of a set of functions on an interval I.

True-False Review

For Questions 1–9, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. Every vector space V possesses a unique minimal spanning set.
2. The set of column vectors of a 5×7 matrix A must be linearly dependent.
3. The set of column vectors of a 7×5 matrix A must be linearly independent.
4. Any nonempty subset of a linearly independent set of vectors is linearly independent.
5. If the Wronskian of a set of functions is nonzero at some point x_0 in an interval I, then the set of functions is linearly independent.
6. If it is possible to express one of the vectors in a set S as a linear combination of the others, then S is a linearly dependent set.
7. If a set of vectors S in a vector space V contains a linearly dependent subset, then S is itself a linearly dependent set.
8. A set of three vectors in a vector space V is linearly dependent if and only if all three vectors are proportional to one another.
9. If the Wronskian of a set of functions is identically zero at every point of an interval I, then the set of functions is linearly dependent.

Problems

For Problems 1–9, determine whether the given set of vectors is linearly independent or linearly dependent in $\mathbb{R}^n$. In the case of linear dependence, find a dependency relationship.

1. $\{(1, -1), (1, 1)\}$.
2. $\{(2, -1), (3, 2), (0, 1)\}$.
3. $\{(1, -1, 0), (0, 1, -1), (1, 1, 1)\}$.
4. $\{(1, 2, 3), (1, -1, 2), (1, -4, 1)\}$.
5. $\{(-2, 4, -6), (3, -6, 9)\}$.
6. $\{(1, -1, 2), (2, 1, 0)\}$.
7. $\{(-1, 1, 2), (0, 2, -1), (3, 1, 2), (-1, -1, 1)\}$.
8. $\{(1, -1, 2, 3), (2, -1, 1, -1), (-1, 1, 1, 1)\}$.
9. $\{(2, -1, 0, 1), (1, 0, -1, 2), (0, 3, 1, 2), (-1, 1, 2, 1)\}$.
10. Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (4, 5, 6)$, $\mathbf{v}_3 = (7, 8, 9)$. Determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent in $\mathbb{R}^3$. Describe
$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$
geometrically.
11. Consider the vectors $\mathbf{v}_1 = (2, -1, 5)$, $\mathbf{v}_2 = (1, 3, -4)$, $\mathbf{v}_3 = (-3, -9, 12)$ in $\mathbb{R}^3$.
    (a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.
    (b) Is $\mathbf{v}_1 \in \text{span}\{\mathbf{v}_2, \mathbf{v}_3\}$? Draw a picture illustrating your answer.
12. Determine all values of the constant k for which the vectors $(1, 1, k)$, $(0, 2, k)$, and $(1, k, 6)$ are linearly dependent in $\mathbb{R}^3$.

For Problems 13–14, determine all values of the constant k for which the given set of vectors is linearly independent in $\mathbb{R}^4$.

13. $\{(1, 0, 1, k), (-1, 0, k, 1), (2, 0, 1, 3)\}$.
14. $\{(1, 1, 0, -1), (1, k, 1, 1), (2, 1, k, 1), (-1, 1, 1, k)\}$.

For Problems 15–17, determine whether the given set of vectors is linearly independent in $M_2(\mathbb{R})$.

15. $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 3 & 6 \\ 0 & 4 \end{bmatrix}$.
16. $A_1 = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$.

17. $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix}$.

For Problems 18–19, determine whether the given set of vectors is linearly independent in P_1.

18. $p_1(x) = 1 - x, \quad p_2(x) = 1 + x.$

19. $p_1(x) = 2 + 3x, \quad p_2(x) = 4 + 6x.$

20. Show that the vectors

$$p_1(x) = a + bx \quad \text{and} \quad p_2(x) = c + dx$$

are linearly independent in P_1 if and only if the constants a, b, c, d satisfy $ad - bc \neq 0$.

21. If $f_1(x) = \cos 2x$, $f_2(x) = \sin^2 x$, $f_3(x) = \cos^2 x$, determine whether $\{f_1, f_2, f_3\}$ is linearly dependent or linearly independent in $C^\infty(-\infty, \infty)$.

For Problems 22–28, determine a linearly independent set of vectors that spans the same subspace of V as that spanned by the original set of vectors.

22. $V = \mathbb{R}^3$, $\{(1, 2, 3), (-3, 4, 5), (1, -\frac{4}{3}, -\frac{5}{3})\}$.

23. $V = \mathbb{R}^3$, $\{(3, 1, 5), (0, 0, 0), (1, 2, -1), (-1, 2, 3)\}$.

24. $V = \mathbb{R}^3$, $\{(1, 1, 1), (1, -1, 1), (1, -3, 1), (3, 1, 2)\}$.

25. $V = \mathbb{R}^4$, $\{(1, 1, -1, 1), (2, -1, 3, 1), (1, 1, 2, 1), (2, -1, 2, 1)\}$.

26. $V = M_2(\mathbb{R})$,

$$\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 5 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \right\}.$$

27. $V = P_1$, $\{2 - 5x, 3 + 7x, 4 - x\}$.

28. $V = P_2$, $\{2 + x^2, 4 - 2x + 3x^2, 1 + x\}$.

For Problems 29–33, use the Wronskian to show that the given functions are linearly independent on the given interval I.

29. $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$, $I = (-\infty, \infty)$.

30. $f_1(x) = \sin x$, $f_2(x) = \cos x$, $f_3(x) = \tan x$, $I = (-\pi/2, \pi/2)$.

31. $f_1(x) = 1$, $f_2(x) = 3x$, $f_3(x) = x^2 - 1$, $I = (-\infty, \infty)$.

32. $f_1(x) = e^{2x}$, $f_2(x) = e^{3x}$, $f_3(x) = e^{-x}$, $I = (-\infty, \infty)$.

33.

$$f_1(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ 3x^3, & \text{if } x < 0, \end{cases}$$

$f_2(x) = 7x^2$, $I = (-\infty, \infty)$.

For Problems 34–36, show that the Wronskian of the given functions is identically zero on $(-\infty, \infty)$. Determine whether the functions are linearly independent or linearly dependent on that interval.

34. $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = 2x - 1$.

35. $f_1(x) = e^x$, $f_2(x) = e^{-x}$, $f_3(x) = \cosh x$.

36. $f_1(x) = 2x^3$,

$$f_2(x) = \begin{cases} 5x^3, & \text{if} x \geq 0, \\ -3x^3, & \text{if } x < 0. \end{cases}$$

37. Consider the functions $f_1(x) = x$,

$$f_2(x) = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

(a) Show that f_2 is not in $C^1(-\infty, \infty)$.

(b) Show that $\{f_1, f_2\}$ is linearly dependent on the intervals $(-\infty, 0)$ and $[0, \infty)$, while it is linearly independent on the interval $(-\infty, \infty)$. Justify your results by making a sketch showing both of the functions.

38. Determine whether the functions $f_1(x) = x$,

$$f_2(x) = \begin{cases} x, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

are linearly dependent or linearly independent on $I = (-\infty, \infty)$.

39. Show that the functions

$$f_1(x) = \begin{cases} x - 1, & \text{if } x \geq 1, \\ 2(x - 1), & \text{if } x < 1, \end{cases}$$

$f_2(x) = 2x$, $f_3(x) = 3$ form a linearly independent set on $(-\infty, \infty)$. Determine all intervals on which $\{f_1, f_2, f_3\}$ is linearly dependent.

40. **(a)** Show that $\{1, x, x^2, x^3\}$ is linearly independent on every interval.

(b) If $f_k(x) = x^k$ for $k = 0, 1, \dots, n$, show that $\{f_0, f_1, \dots, f_n\}$ is linearly independent on every interval for all fixed n.

41. **(a)** Show that the functions

$$f_1(x) = e^{r_1 x}, \quad f_2(x) = e^{r_2 x}, \quad f_3(x) = e^{r_3 x}$$

have Wronskian

$$W[f_1, f_2, f_3](x) = e^{(r_1+r_2+r_3)x}\begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix}$$
$$= e^{(r_1+r_2+r_3)x}(r_3 - r_1)(r_3 - r_2)(r_2 - r_1),$$

and hence determine the conditions on r_1, r_2, r_3 such that $\{f_1, f_2, f_3\}$ is linearly independent on every interval.

(b) More generally, show that the set of functions

$$\{e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}\}$$

is linearly independent on every interval if and only if all of the r_i are distinct. [**Hint:** Show that the Wronskian of the given functions is a multiple of the $n \times n$ Vandermonde determinant, and then use Problem 21 in Section 3.3.]

42. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a linearly independent set in a vector space V, and let $\mathbf{v} = \alpha\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w} = \mathbf{v}_1 + \alpha\mathbf{v}_2$, where α is a constant. Use Definition 4.5.4 to determine all values of α for which $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent.

43. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in a vector space V, and $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are each linear combinations of them, prove that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent.

44. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ be a set of linearly independent vectors in a vector space V and suppose that the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are each linear combinations of them. It follows that we can write

$$\mathbf{u}_k = \sum_{i=1}^{m} a_{ik}\mathbf{v}_i, \qquad k = 1, 2, \dots, n,$$

for appropriate constants a_{ik}.

(a) If $n > m$, prove that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is linearly dependent on V.

(b) If $n = m$, prove that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is linearly independent in V if and only if $\det[a_{ij}] \neq 0$.

(c) If $n < m$, prove that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is linearly independent in V if and only if $\operatorname{rank}(A) = n$, where $A = [a_{ij}]$.

(d) Which result from this section do these results generalize?

45. Prove from the definition of "linearly independent" that if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent and if A is an invertible $n \times n$ matrix, then the set $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$ is linearly independent.

46. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and $\mathbf{v}_3$ is not in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

47. Generalizing the previous exercise, prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly independent and $\mathbf{v}_{k+1}$ is not in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k+1}\}$ is linearly independent.

48. Prove Theorem 4.5.2.

49. Prove Proposition 4.5.7.

50. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ spans a vector space V, then for every vector $\mathbf{v}$ in V, $\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly dependent.

51. Prove that if $V = P_n$ and $S = \{p_1, p_2, \dots, p_k\}$ is a set of vectors in V each of a different degree, then S is linearly independent. [**Hint:** Assume without loss of generality that the polynomials are ordered in descending degree: $\deg(p_1) > \deg(p_2) > \cdots > \deg(p_k)$. Assuming that $c_1 p_1 + c_2 p_2 + \cdots + c_k p_k = 0$, first show that c_1 is zero by examining the highest degree. Then repeat for lower degrees to show successively that $c_2 = 0$, $c_3 = 0$, and so on.]

4.6 Bases and Dimension

The results of the previous section show that if a minimal spanning set exists in a (nontrivial) vector space V, it cannot be linearly dependent. Therefore if we are looking for minimal spanning sets for V, we should focus our attention on spanning sets that are linearly independent. One of the results of this section establishes that *every* spanning set for V that is linearly independent is indeed a minimal spanning set. Such a set will be