Introduction to Fourier Series

MA 16021

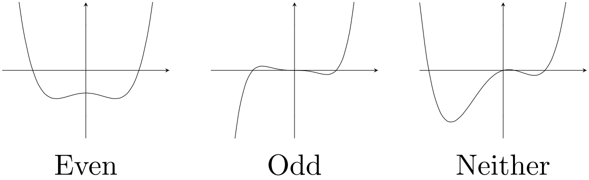
October 15, 2014

Even and odd functions

Definition

A function $f(x)$ is said to be *even* if $f(-x) = f(x)$.
The function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$.

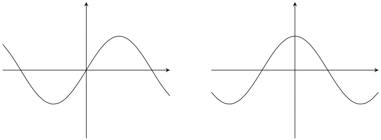
Graphically, even functions have symmetry about the y-axis, whereas odd functions have symmetry around the origin.



Even and odd functions

Examples:

- Sums of odd powers of x are odd: $5x^3 - 3x$
- Sums of even powers of x are even: $-x^6 + 4x^4 + x^2 - 3$
- $\sin x$ is odd, and $\cos x$ is even



$\sin x$ (odd) $\cos x$ (even)

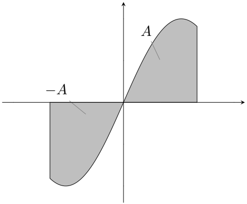
- The product of two odd functions is even: $x \sin x$ is even
- The product of two even functions is even: $x^2 \cos x$ is even
- The product of an even function and an odd function is odd: $\sin x \cos x$ is odd

Integrating odd functions over symmetric domains

Let $p > 0$ be any fixed number. If $f(x)$ is an odd function, then

$$\int_{-p}^{p} f(x)\ dx = 0.$$

Intuition: The area beneath the curve on $[-p, 0]$ is the same as the area under the curve on $[0, p]$, but opposite in sign. So, they cancel each other out!

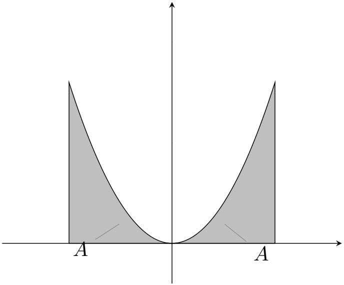


Integrating even functions over symmetric domains

Let $p > 0$ be any fixed number. If $f(x)$ is an even function, then

$$\int_{-p}^{p} f(x)\ dx = 2\int_{0}^{p} f(x)\ dx.$$

Intuition: The area beneath the curve on $[-p, 0]$ is the same as the area under the curve on $[0, p]$, but this time with the same sign. So, you can just find the area under the curve on $[0, p]$ and double it!



Periodic functions

Definition

A function $f(x)$ is said to be *periodic* if there exists a number $T > 0$ such that $f(x+T) = f(x)$ for every x. The smallest such T is called the *period* of $f(x)$.

Intiutively, periodic functions have repetitive behavior.
A periodic function can be defined on a finite interval, then copied and pasted so that it repeats itself.
Examples

- $\sin x$ and $\cos x$ are periodic with period 2π
- $\sin(\pi x)$ and $\cos(\pi x)$ are periodic with period 2
- If L is a fixed number, then $\sin(\frac{2\pi x}{L})$ and $\cos(\frac{2\pi x}{L})$ have period L

Sine and cosine are the most “basic” periodic functions!

Fourier series

Let $p > 0$ be a fixed number and $f(x)$ be a periodic function with period $2p$, defined on $(-p, p)$. The Fourier series of $f(x)$ is a way of expanding the function $f(x)$ into an infinite series involving sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{p}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{p}) \tag{2.1}$$

where a_0, a_n, and b_n are called the Fourier coefficients of $f(x)$, and are given by the formulas

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\, dx, \qquad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos(\frac{n\pi x}{p})\, dx, \tag{2.2}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin(\frac{n\pi x}{p})\, dx,$$

Fourier Series

Remarks:

- To find a Fourier series, it is sufficient to calculate the integrals that give the coefficients a_0, a_n, and b_n and plug them in to the big series formula, equation (2.1) above.
- Typically, $f(x)$ will be piecewise defined.
- Big advantage that Fourier series have over Taylor series: the function $f(x)$ can have discontinuities!

Useful identities for Fourier series: if n is an integer, then

- $\sin(n\pi) = 0$
  e.g. $\sin(\pi) = \sin(2\pi) = \sin(3\pi) = \sin(20\pi) = 0$
- $\cos(n\pi) = (-1)^n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$

$$\text{e.g. } \cos(\pi) = \cos(3\pi) = \cos(5\pi) = -1,$$
$$\text{but } \cos(0\pi) = \cos(2\pi) = \cos(4\pi) = 1.$$

Fourier coefficients of an even function

If $f(x)$ is an even function, then the formulas for the coefficients simplify. Specifically, since $f(x)$ is even, $f(x)\sin(\frac{n\pi x}{p})$ is an odd function, and thus

$$b_n = \frac{1}{p}\int_{-p}^{p} \overbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\sin(\frac{n\pi x}{p})}_{\text{odd}}}^{\text{odd}}\, dx = 0$$

Therefore, for even functions, you can automatically conclude (no computations necessary!) that the b_n coefficients are all 0.

Fourier coefficients for an odd function

If $f(x)$ is odd, then we get two freebies:

$$a_0 = \frac{1}{p}\int_{-p}^{p} \overbrace{f(x)}^{\text{odd}}\ dx = 0$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \overbrace{\underbrace{f(x)}_{\text{odd}}\,\underbrace{\cos(\frac{n\pi x}{p})}_{\text{even}}}^{\text{odd}}\ dx = 0$$

Note: In general, your function may be neither even nor odd. In those cases, you should use the original formulas for computing Fourier coefficients, given in equation (2.2).

Disclaimer

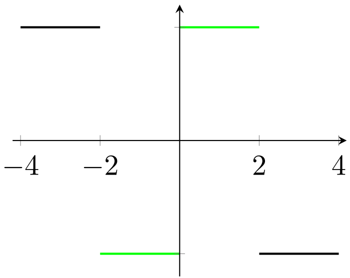
The following examples are just meant to give you an idea of what sorts of computations are involved in finding a Fourier series. You're not meant to be able to carry out these computations yet. So just sit back, relax, and enjoy the ride!

Example 1

Let $f(x)$ be periodic and defined on one period by the formula

$$f(x) = \begin{cases} -1 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$$

Graph of $f(x)$ (original part in green):



Example 1

Since $f(x)$ is an odd function, we conclude that $a_0 = a_n = 0$ for each n. A bit of computation reveals

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{n\pi x}{2})\;dx = \frac{2}{n\pi}(1-\cos(n\pi)) = \frac{2}{n\pi}(1-(-1)^n)$$

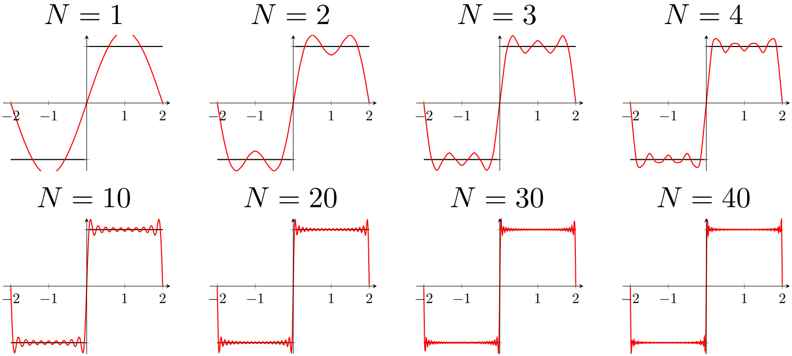
Therefore

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-(-1)^n)\sin(\frac{n\pi x}{2}) \\ &= \underbrace{\frac{4}{\pi}\sin(\frac{\pi x}{2})}_{n=1} + \underbrace{\frac{4}{3\pi}\sin(\frac{3\pi x}{2})}_{n=3} + \cdots \end{aligned}$$

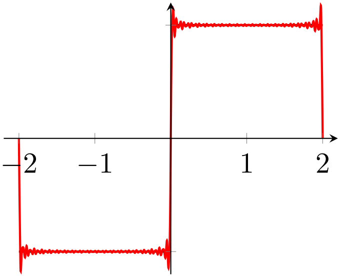
Notice: The even b_n terms are all 0 since $1-(-1)^n = 1-1 = 0$ when n is even.

Example 1

If we plot the first N non-zero terms, we get approximations of $f(x)$:



Example 1



Observations:

- As the number of terms used increases, the approximation gets closer and closer to the original function
- The original function has a discontinuity at $x = 0$. The approximation converges to 0 there, which is the average of the right- and left-hand limits as $x \to 0$.

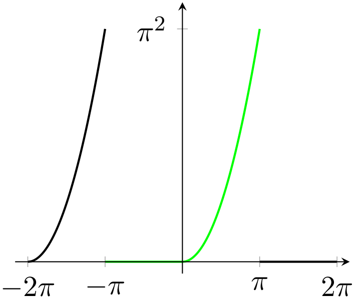
In general, if $f(x)$ has a discontinuity at x_0, then the Fourier series converges to the average of $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$.

Example 2

Let $f(x)$ be periodic and defined on one period by the formula

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 < x < \pi \end{cases}$$

Graph of $f(x)$ (original part in green):



The function is neither even nor odd since it has no symmetry.

Example 2

After some calculations (which are very tedious and involve lots of IBP),

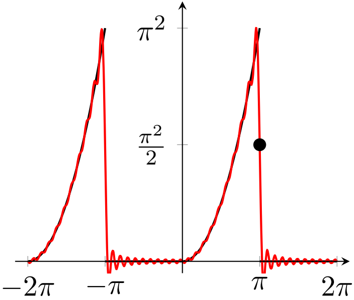
$$a_0 = \frac{1}{3}\pi^2, \qquad a_n = \frac{2(-1)^n}{n^2}, \qquad b_n = \frac{(-1)^n(2-\pi^2 n^2)-2}{n^3\pi}$$

Thus,

$$f(x) = \underbrace{\frac{1}{6}\pi^2}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty}\left(\underbrace{\frac{2(-1)^n}{n^2}}_{a_n}\cos(nx) + \underbrace{\frac{(-1)^n(2-n^2\pi^2)-2}{n^3\pi}}_{b_n}\sin(nx)\right)$$

Example 2

Plot of Fourier series (first 20 terms):



Notice: At $x = \pi$, the series converges to $\frac{1}{2}(\pi^2 + 0) = \frac{\pi^2}{2}$.

Example 2

By plugging in $x = \pi$ into the Fourier series for $f(x)$ and using the fact that the series converges to $\frac{\pi^2}{2}$,

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left(\frac{2(-1)^n}{n^2} \cos(n\pi) + \frac{(-1)^n(2 - \pi^2 n^2) - 2}{n^3 \pi} \sin(n\pi) \right)$$

Because $\sin(n\pi) = 0$ and $(-1)^n \cos(n\pi) = (-1)^n(-1)^n = 1$, one can derive the following formula (c.f. example from lecture 14)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

That's all for now! Reminders:

- Review Friday
- Next office hours: Thursday, 6:00 – 7:00 pm (Math 609)
- Exam 2: Monday, 6:30 pm in Elliot