Lesson 35

Basic Power Functions:

- $f(x) = bx^a$
  - the base x is a variable and the exponent a is a constant, b is a non-zero constant coefficient
- Examples:
  - $f(x) = x^2$
  - $g(x) = 3x^{\frac{1}{2}}$
  - $h(x) = \frac{1}{2}x^{-1}$
- there are no restrictions on the exponent a, but there could be restrictions the base x depending on the value of exponent
  - for the function $g(x) = x^{\frac{1}{2}}$, restrictions: $x \geq 0$
  - for the function $h(x) = x^{-1}$, restrictions: $x \neq 0$

Basic Exponential Functions:

- $f(x) = ba^x$
  - the exponent $\boldsymbol{x}$ **is a variable** and **base $\boldsymbol{a}$ is a constant within the guidelines below**, b is a non-zero constant
- there are no restrictions on the exponent x, but there are restrictions on the base a
  - a cannot be negative $(a \geq 0)$ because x could be a fraction $\left(x = \frac{1}{2}\right)$ for example, $(-2)^{1/2}$ is not defined

  - a cannot be zero $(a \neq 0)$ because x could be negative $(x = -1)$ 0^{-1} is not defined

  - a cannot be one $(a \neq 1)$ because $f(x) = a^x$ would not be $1 - 1$ $f(x) = 1^x$ is the same as $f(x) = 1$, a horizontal line

An exponential function $f(x) = a^x$ must have a positive base other than 1 $(a > 0\ and\ a \neq 1)$.

Example 1: Find an exponential function of the form $f(x) = b \cdot a^x$, given the following information, then find the domain and range of each function.

a. passes through the points $P(0, 1)$ and $Q(-3, 8)$

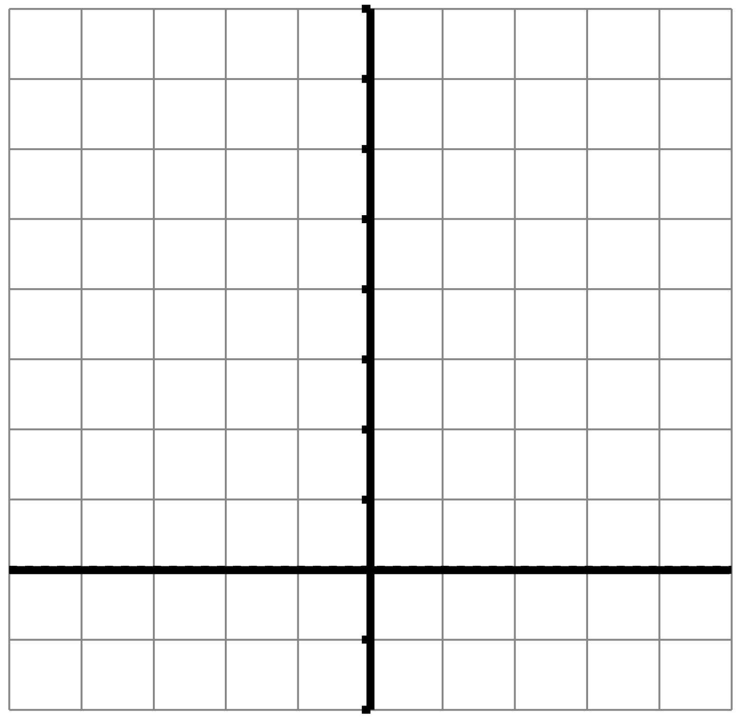
b. passes through the points $P(0, -2)$ and $Q(1, -4)$

c. y-intercept is 6; passes through the point $P\left(2, \frac{3}{32}\right)$

Because the exponent x is unrestricted, **the domain of an exponential function is all real numbers** $(-\infty, \infty)$. The range however is restricted; as long as $a > 0$, a^x will always produce positive outputs, so **the range of the exponential function** $\boldsymbol{f(x) = a^x}$ **is only positive numbers** $(\mathbf{0}, \infty)$. The range could change if an exponential function is transformed (shifted and/or reflected vertically).

Example 2: Sketch the graph of $f(x) = 2^x$, then find the following:

Inputs	**Outputs**
x	$f(x) = 2^x$
$x \to -\infty$	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
$x \to \infty$	



Domain:

Range:

$f(x) = 0$ when $x =$

x-intercepts:

$f(x) > 0$:

$f(0) =$

$f(x) < 0$:

y-intercepts:

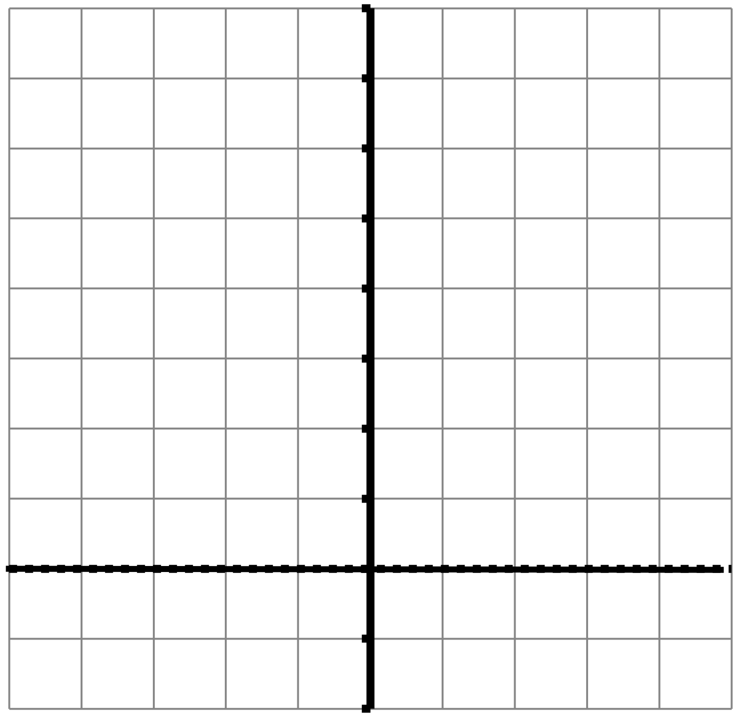
Increasing:

Decreasing:

$f(x)$ is even/odd/neither: $\quad f(-1) = \quad f(1) =$

<u>Example 2 ½ :</u> Sketch the graph of $f(x) = \left(\frac{1}{2}\right)^x$, then find the following:

<u>Inputs</u>	**<u>Outputs</u>**
x	$f(x) = \left(\frac{1}{2}\right)^x$
$x \to -\infty$	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
$x \to \infty$	



Domain:

Range:

$f(x) = 0$ when $x =$

x-intercepts:

$f(x) > 0$:

$f(0) =$

$f(x) < 0$:

y-intercepts:

Increasing:

Decreasing:

$f(x)$ is even/odd/neither: $f(-1) =$ $f(1) =$

Asymptotes:

Horizontal Asymptote:

- a horizontal line ($y = \#$) that the graph of a function approaches when the inputs are large positive numbers ($x \to \infty$) or large negative numbers ($x \to -\infty$)
    - in the case of $f(x) = 2^x$, as the inputs get smaller and smaller ($x \to -\infty$), the outputs get closer and closer to zero ($f(x) \to 0$), so the graph has a horizontal asymptote at $y = 0$
    - the graph of every exponential function will have a horizontal asymptote
- in WebAssign, horizontal asymptotes will not be dotted lines when they lie on the x-axis, but only when they are above or below the x-axis

Example 3: Re-write each of the following functions in terms of $f(x) = 2^x$, then match the transformation with the appropriate graph on page 6.

a. $g(x) = -2^x$

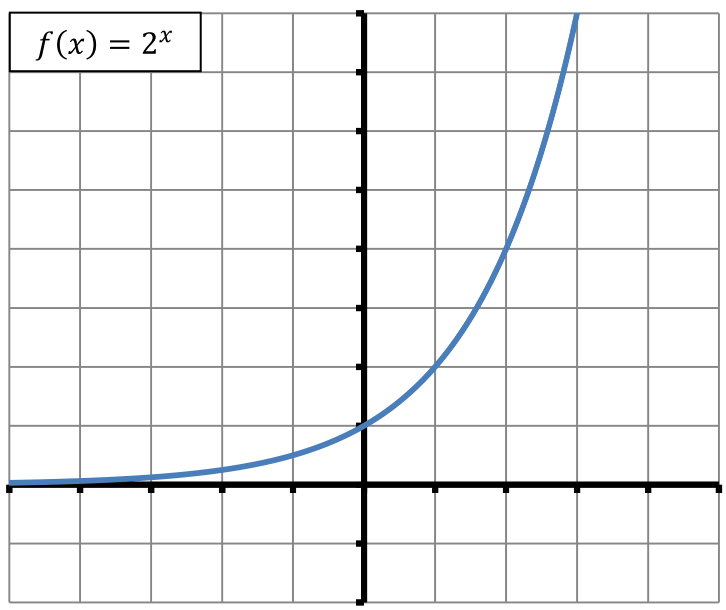
b. $h(x) = 2^{-x}$

c. $j(x) = 2^{x-3}$

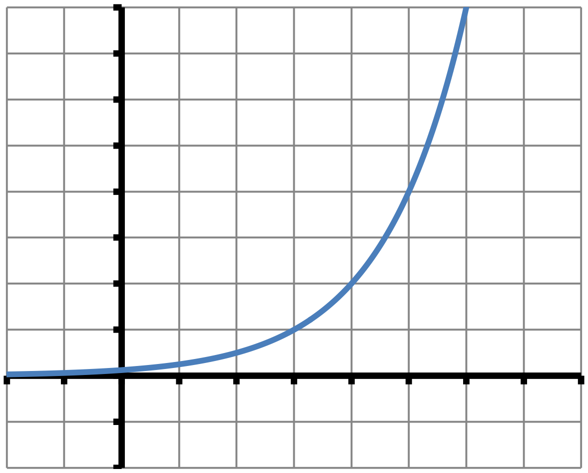
d. $k(x) = 2^x - 3$

e. $m(x) = 3(2^x)$

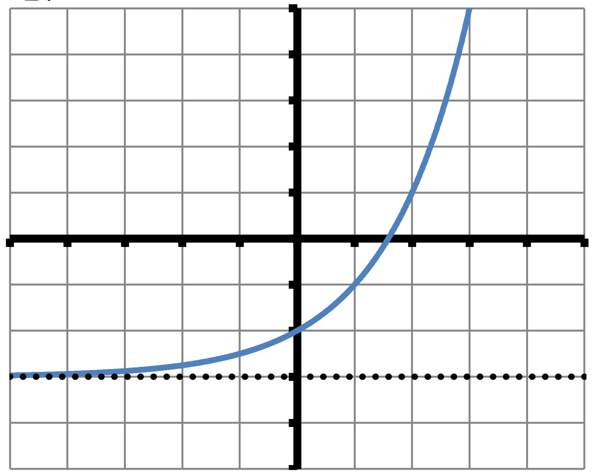
f. $n(x) = 2^{3x}$



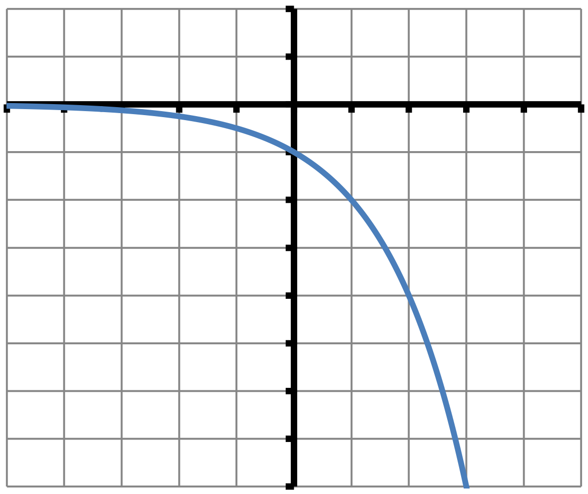
A.



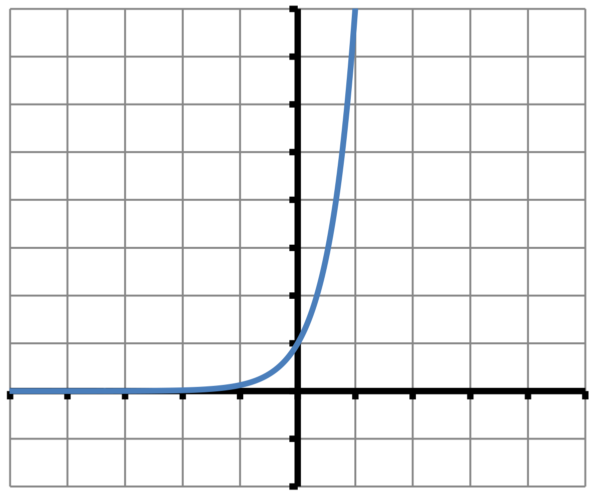
B.



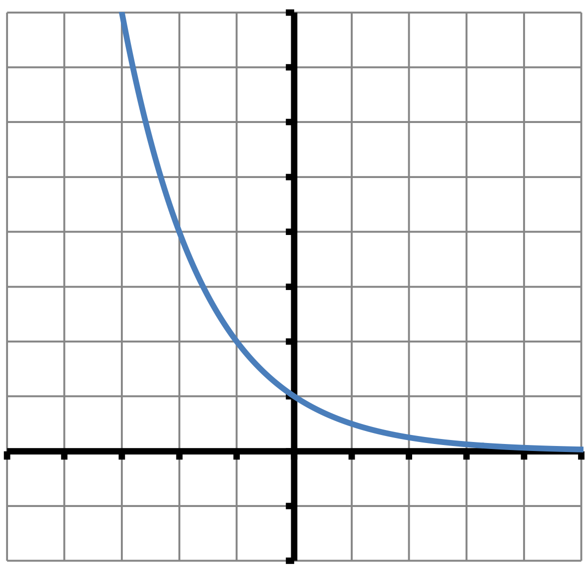
C.



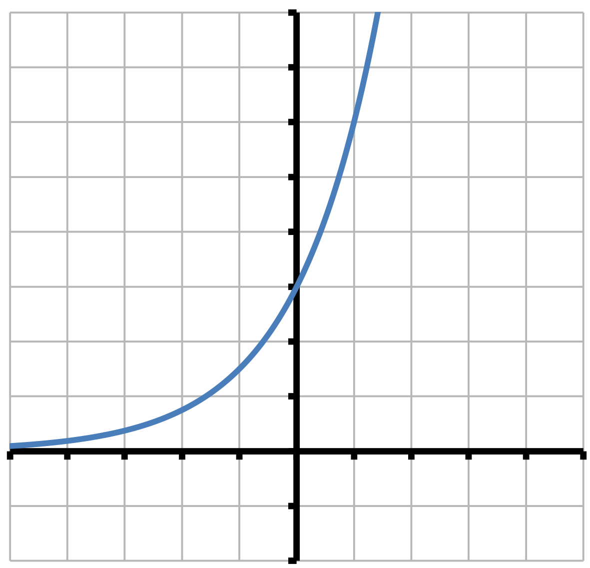
D.



E.



F.



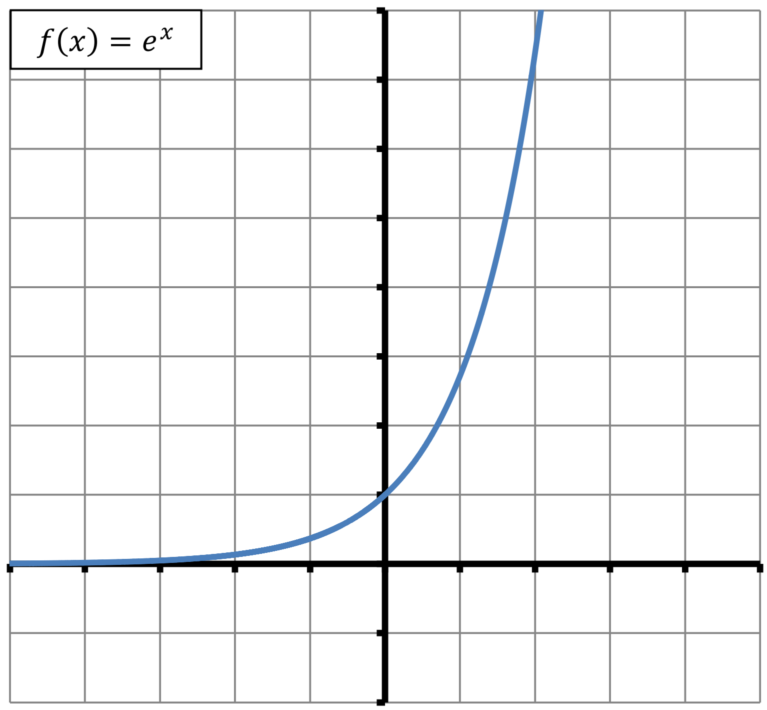
When solving problems like this on the homework, you can either use transformations (like I did) or you can make new input/output tables for each new function to sketch/identify the graphs. Transformations will most likely be quicker, but it is your choice.

Natural Exponential Functions:

- $f(x) = e^x$, where the exponent x is a variable and base e is the number 2.71828…
- the base e is **NOT** a variable, it is simply a number (like π or i)
- the domain is still unrestricted $(-\infty, \infty)$ and the range is still only positive numbers $(0, \infty)$

Example 4: Re-write each of the following functions below in terms of $f(x) = e^x$, then match the transformation with the appropriate graph on page 8.

Inputs	**Outputs**
x	$f(x) = e^x$
$x \to -\infty$	$f(x) \to 0$
-2	$\frac{1}{e^2} \approx 0.14$
-1	$\frac{1}{e} \approx 0.37$
0	1
1	$e \approx 2.72$
2	$e^2 \approx 7.39$
$x \to \infty$	$f(x) \to \infty$



a. $g(x) = e^{-x}$

b. $h(x) = -e^x$

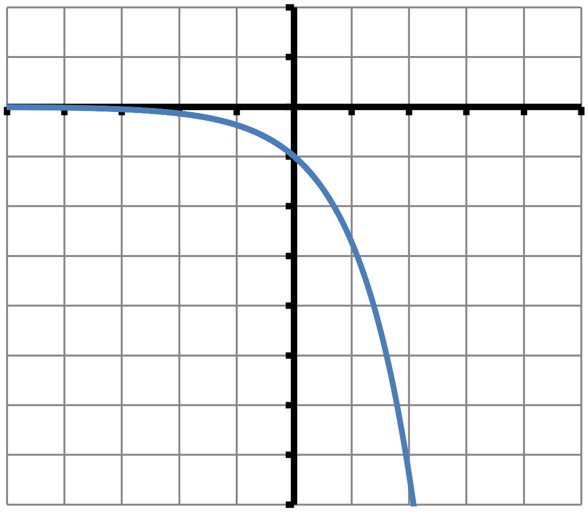
c. $j(x) = e^{x+2}$

d. $k(x) = e^x + 2$

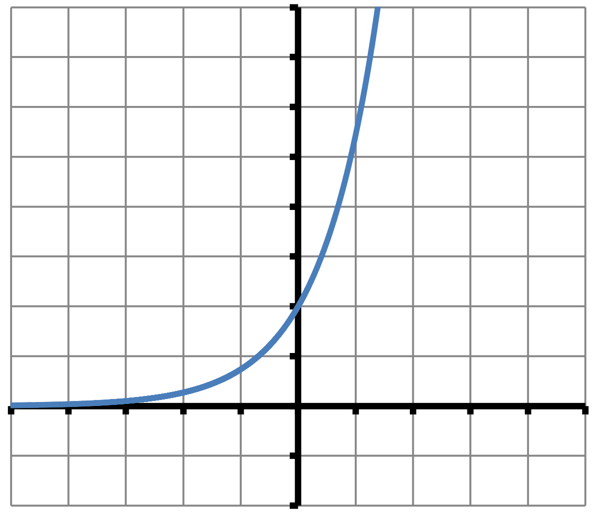
e. $m(x) = 2e^x$

f. $n(x) = e^{2x}$

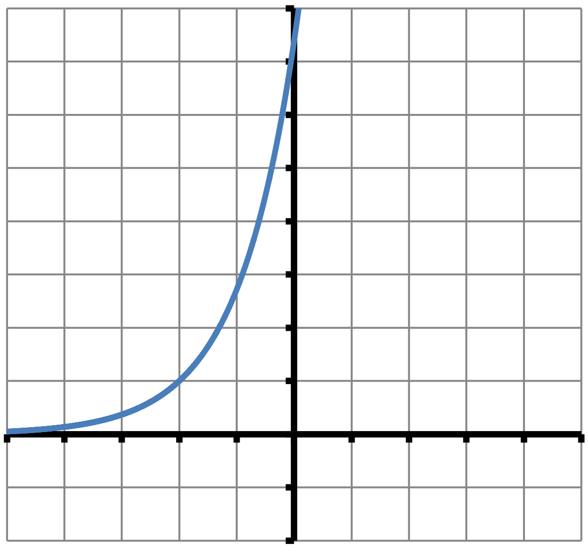
A.



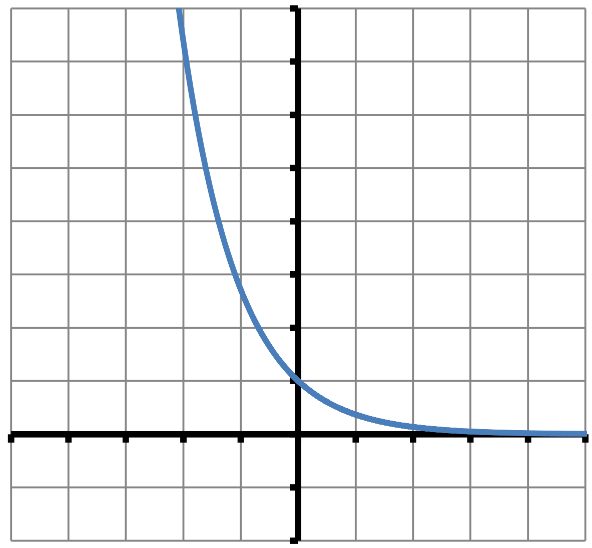
B.



C.



D.



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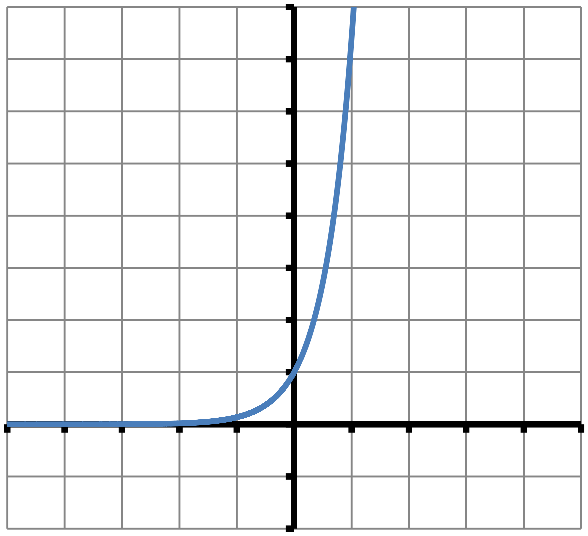

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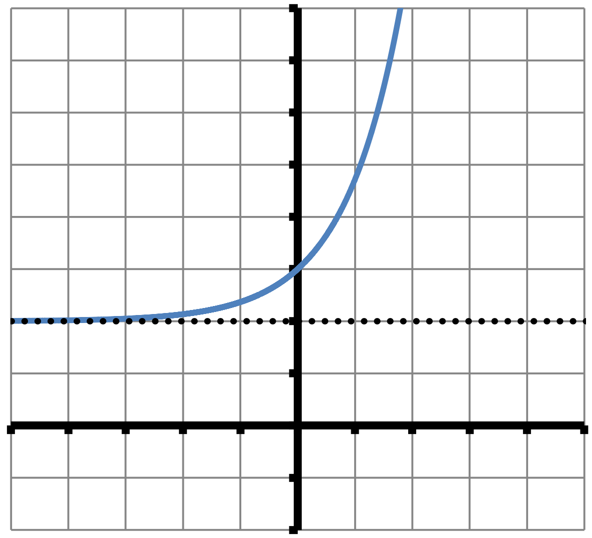

Again, when solving problems like this on the homework you can use any method you like to sketch/identify the graphs of the functions.
All the transformations discussed in Lessons 22 & 23 hold true for exponential functions (shifting, stretching, compressing, and reflecting). The only way for an exponential function to be equal to zero is if it is transformed in a way that it crosses the x-axis (shifted and/or reflected vertically).

if $f(x) = 2^x$, then $f(x) \neq 0$
if $g(x) = 2^x - 4$, then $g(x) = 0$ when $x = 2$

<u>Example 5:</u> Find the zeros (numbers where x-intercepts are located) of the following functions.

a. $g(x) = x2^x + 2^x$

$g(x) = 0$ when $x =$

b. $h(x) = -x^2 2^{-x} + 2x2^{-x}$

$h(x) = 0$ when $x =$

c. $j(x) = x^3(4e^{4x}) + 3x^2e^{4x}$

$j(x) = 0$ when $x =$

d. $k(x) = x^2e^{2x} + 5xe^{2x} + 6e^{2x}$

$k(x) = 0$ when $x =$

Referring back to the exponential graphs from Examples 2, 3, & 4, notice that all the graphs of exponential functions are either increasing or decreasing; there are no turning points. Based on the Theorem on Increasing/Decreasing Functions from Lesson 34, this means that exponential functions are one-to-one.

Theorem on Exponential Functions:

- exponential functions are one-to-one, so if a base to power is equal to the same base to another power, the powers must be the same
  - if $2^x = 2^y$, then $x = y$
  - if $2^x = 8^y$, then $x = 3y$ because $2^x = 2^{3y}$
- if two exponential expressions are equal, and the bases are same, the exponents must be the equal

Example 6: Solve each of the following equations by making the bases the same and simplifying, then setting the exponents equal to each other.

a. $7^{x+6} = 7^{3x-4}$

b. $e^{(x^2)} = e^{7x-12}$

c. $2^{2x-6} = 8^{4-x}$

d. $27^{x-1} = 9^{2x-3}$

One application of exponential functions is compound interest, which is when interest is calculated on the total value of a sum, not just on the principal. We will look at two ways to calculate compound interest; n times per year or continuously.

Compound Interest Formulas:

- when interest is compounded n times per year, we use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where n is the number of compounding periods
- when interest is compounded continuously, we use the formula $A = Pe^{rt}$
- in both cases:
  - A is the accumulated value of the investment
  - P is the principal (the original amount invested)
  - r is the annual interest rate as a decimal
  - t is the number years the principal is invested
- these formulas will be provided on the Final Exam, if needed

Example 7: If $1000 is invested at a rate of 7% per year compounded monthly, find value of the investment at each given time and round to the nearest cent.

a) 1 month

b. 6 months

c. 1 year

d. 20 years

Example 8: Parents of a newborn baby are given a gift of \$20,000 and will choose between two options to invest for their child's college fund. Option 1 is to invest the gift in a fund that pays an average annual interest rate of 8% compounded semiannually; option 2 is to invest the gift in a fund that pays an average annual interest rate of 7.75% compounded continuously. Assuming each investment has a term of 18 years, which is the better option for the parents?

What if the rates are the same?

Example 9: A proposed alternative to the current Social Security system is to set-up an account with \$10,000 for every child born in the United States to parents who are U.S. citizens. The account would be payable on the newborns 70^{th} birthday. Calculate the amount of each account if the average annual interest is 5% compounded:

a. Annually b. Quarterly c. Continuously