(a) Show that Eq. (i) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)};$$ (ii)

thus Eq. (i) is homogeneous.

(b) Introduce a new dependent variable v so that $v = y/x$, or $y = xv(x)$. Express dy/dx in terms of x, v, and dv/dx.

(c) Replace y and dy/dx in Eq. (ii) by the expressions from part (b) that involve v and dv/dx. Show that the resulting differential equation is

$$v + x\frac{dv}{dx} = \frac{v - 4}{1 - v},$$

or

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}.$$ (iii)

Observe that Eq. (iii) is separable.

(d) Solve Eq. (iii) for v in terms of x.

(e) Find the solution of Eq. (i) by replacing v by y/x in the solution in part (d).

(f) Draw a direction field and some integral curves for Eq. (i). Recall that the right side of Eq. (1) actually depends only on the ratio y/x. This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 30 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In each of Problems 31 through 38:

(a) Show that the given equation is homogeneous.

(b) Solve the differential equation.

(c) Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

▶ 31. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$

▶ 32. $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$

▶ 33. $\dfrac{dy}{dx} = \dfrac{4y - 3x}{2x - y}$

▶ 34. $\dfrac{dy}{dx} = -\dfrac{4x + 3y}{2x + y}$

▶ 35. $\dfrac{dy}{dx} = \dfrac{x + 3y}{x - y}$

▶ 36. $(x^2 + 3xy + y^2)\,dx - x^2dy = 0$

▶ 37. $\dfrac{dy}{dx} = \dfrac{x^2 - 3y^2}{2xy}$

▶ 38. $\dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$

2.3 Modeling with First Order Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and

their solutions lead to equations relating the variables and parameters in the problem. These equations often enable you to make predictions about how the natural process will behave in various circumstances. It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive in an experimental setting. Nevertheless, mathematical modeling and experiment or observation are both critically important and have somewhat complementary roles in scientific investigations. Mathematical models are validated by comparison of their predictions with experimental results. On the other hand, mathematical analyses may suggest the most promising directions to explore experimentally, and may indicate fairly precisely what experimental data will be most helpful.

In Sections 1.1 and 1.2 we formulated and investigated a few simple mathematical models. We begin by recapitulating and expanding on some of the conclusions reached in those sections. Regardless of the specific field of application, there are three identifiable steps that are always present in the process of mathematical modeling.

Construction of the Model.   This involves a translation of the physical situation into mathematical terms, often using the steps listed at the end of Section 1.1. Perhaps most critical at this stage is to state clearly the physical principle(s) that are believed to govern the process. For example, it has been observed that in some circumstances heat passes from a warmer to a cooler body at a rate proportional to the temperature difference, that objects move about in accordance with Newton's laws of motion, and that isolated insect populations grow at a rate proportional to the current population. Each of these statements involves a rate of change (derivative) and consequently, when expressed mathematically, leads to a differential equation. The differential equation is a mathematical model of the process.

It is important to realize that the mathematical equations are almost always only an approximate description of the actual process. For example, bodies moving at speeds comparable to the speed of light are not governed by Newton's laws, insect populations do not grow indefinitely as stated because of eventual limitations on their food supply, and heat transfer is affected by factors other than the temperature difference. Alternatively, one can adopt the point of view that the mathematical equations exactly describe the operation of a simplified physical model, which has been constructed (or conceived of) so as to embody the most important features of the actual process. Sometimes, the process of mathematical modeling involves the conceptual replacement of a discrete process by a continuous one. For instance, the number of members in an insect population changes by discrete amounts; however, if the population is large, it seems reasonable to consider it as a continuous variable and even to speak of its derivative.

Analysis of the Model.   Once the problem has been formulated mathematically, one is often faced with the problem of solving one or more differential equations or, failing that, of finding out as much as possible about the properties of the solution. It may happen that this mathematical problem is quite difficult and, if so, further approximations may be indicated at this stage to make the problem mathematically tractable. For example, a nonlinear equation may be approximated by a linear one, or a slowly varying coefficient may be replaced by a constant. Naturally, any such approximations must also be examined from the physical point of view to make sure that the simplified mathematical problem still reflects the essential features of the physical process under investigation. At the same time, an intimate knowledge of the physics of the problem may suggest reasonable mathematical approximations

that will make the mathematical problem more amenable to analysis. This interplay of understanding of physical phenomena and knowledge of mathematical techniques and their limitations is characteristic of applied mathematics at its best, and is indispensable in successfully constructing useful mathematical models of intricate physical processes.

Comparison with Experiment or Observation.   Finally, having obtained the solution (or at least some information about it), you must interpret this information in the context in which the problem arose. In particular, you should always check that the mathematical solution appears physically reasonable. If possible, calculate the values of the solution at selected points and compare them with experimentally observed values. Or, ask whether the behavior of the solution after a long time is consistent with observations. Or, examine the solutions corresponding to certain special values of parameters in the problem. Of course, the fact that the mathematical solution appears to be reasonable does not guarantee it is correct. However, if the predictions of the mathematical model are seriously inconsistent with observations of the physical system it purports to describe, this suggests that either errors have been made in solving the mathematical problem, or the mathematical model itself needs refinement, or observations must be made with greater care.

The examples in this section are typical of applications in which first order differential equations arise.

EXAMPLE
1

Mixing

At time $t = 0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water; see Figure 2.3.1. Assume that water containing $\frac{1}{4}$ lb of salt/gal is entering the tank at a rate of r gal/min, and that the well-stirred mixture is draining from the tank at the same rate. Set up the initial value problem that describes this flow process. Find the amount of salt $Q(t)$ in the tank at any time, and also find the limiting amount Q_L that is present after a very long time. If $r = 3$ and $Q_0 = 2Q_L$, find the time T after which the salt level is within 2% of Q_L. Also find the flow rate that is required if the value of T is not to exceed 45 min.
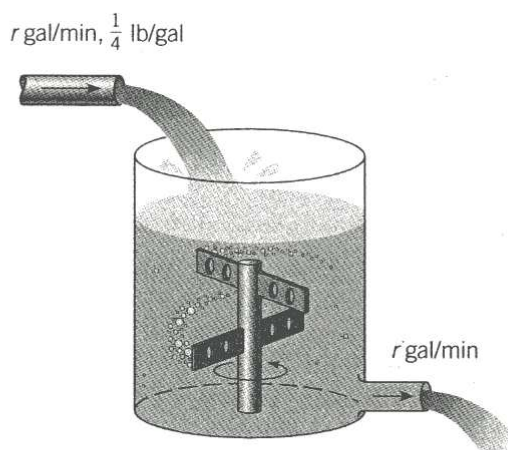
r gal/min, $\frac{1}{4}$ lb/gal

r gal/min

FIGURE 2.3.1   The water tank in Example 1.

We assume that salt is neither created nor destroyed in the tank. Therefore variations in the amount of salt are due solely to the flows in and out of the tank. More precisely, the rate of change of salt in the tank, dQ/dt, is equal to the rate at which salt is flowing in minus the rate at which it is flowing out. In symbols,

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out.} \tag{1}$$

The rate at which salt enters the tank is the concentration $\frac{1}{4}$ lb/gal times the flow rate r gal/min, or $(r/4)$ lb/min. To find the rate at which salt leaves the tank we need to multiply the concentration of salt in the tank by the rate of outflow, r gal/min. Since the rates of flow in and out are equal, the volume of water in the tank remains constant at 100 gal, and since the mixture is "well-stirred," the concentration throughout the tank is the same, namely, $[Q(t)/100]$ lb/gal. Therefore the rate at which salt leaves the tank is $[rQ(t)/100]$ lb/min. Thus the differential equation governing this process is

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}. \tag{2}$$

The initial condition is

$$Q(0) = Q_0. \tag{3}$$

Upon thinking about the problem physically, we might anticipate that eventually the mixture originally in the tank will be essentially replaced by the mixture flowing in, whose concentration is $\frac{1}{4}$ lb/gal. Consequently, we might expect that ultimately the amount of salt in the tank would be very close to 25 lb. We can reach the same conclusion from a geometrical point of view by drawing a direction field for Eq. (2) for any positive value of r.

To solve the problem analytically note that Eq. (2) is both linear and separable. Rewriting it in the usual form for a linear equation, we have

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}. \tag{4}$$

Thus the integrating factor is $e^{rt/100}$ and the general solution is

$$Q(t) = 25 + ce^{-rt/100}, \tag{5}$$

where c is an arbitrary constant. To satisfy the initial condition (3) we must choose $c = Q_0 - 25$. Therefore the solution of the initial value problem (2), (3) is

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100} \tag{6}$$

or

$$Q(t) = 25(1 - e^{-rt/100}) + Q_0 e^{-rt/100}. \tag{7}$$

From Eq. (6) or (7), you can see that $Q(t) \to 25$ (lb) as $t \to \infty$, so the limiting value Q_L is 25, confirming our physical intuition. In interpreting the solution (7), note that the second term on the right side is the portion of the original salt that remains at time t, while the first term gives the amount of salt in the tank due to the action of the flow processes. Plots of the solution for $r = 3$ and for several values of Q_0 are shown in Figure 2.3.2.
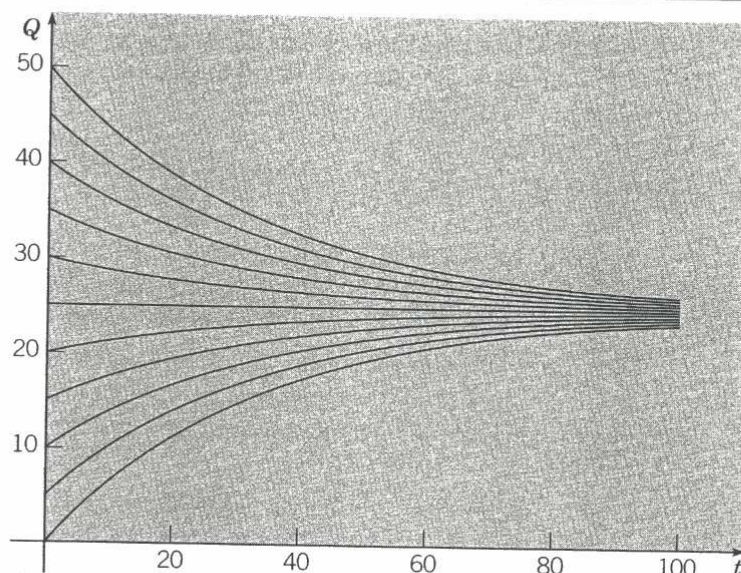
FIGURE 2.3.2 Solutions of the initial value problem (2), (3) for $r = 3$ and several values of Q_0.

Now suppose that $r = 3$ and $Q_0 = 2Q_L = 50$; then Eq. (6) becomes

$$Q(t) = 25 + 25e^{-0.03t}. \tag{8}$$

Since 2% of 25 is 0.5, we wish to find the time T at which $Q(t)$ has the value 25.5. Substituting $t = T$ and $Q = 25.5$ in Eq. (8) and solving for T, we obtain

$$T = (\ln 50)/0.03 \cong 130.4 \text{ (min)}. \tag{9}$$

To determine r so that $T = 45$, return to Eq. (6), set $t = 45$, $Q_0 = 50$, $Q(t) = 25.5$, and solve for r. The result is

$$r = (100/45) \ln 50 \cong 8.69 \text{ gal/min}. \tag{10}$$

Since this example is hypothetical, the validity of the model is not in question. If the flow rates are as stated, and if the concentration of salt in the tank is uniform, then the differential equation (1) is an accurate description of the flow process. While this particular example has no special significance, it is important that models of this kind are often used in problems involving a pollutant in a lake, or a drug in an organ of the body, for example, rather than a tank of salt water. In such cases the flow rates may not be easy to determine, or may vary with time. Similarly, the concentration may be far from uniform in some cases. Finally, the rates of inflow and outflow may be different, which means that the variation of the amount of liquid in the problem must also be taken into account.

EXAMPLE 2

Compound Interest

Suppose that a sum of money is deposited in a bank or money fund that pays interest at an annual rate r. The value $S(t)$ of the investment at any time t depends on the frequency with which interest is compounded as well as the interest rate. Financial institutions have various policies concerning compounding: some compound monthly, some weekly, some even daily. If we assume that compounding takes place *continuously*, then we can set up a simple initial value problem that describes the growth of the investment.

The rate of change of the value of the investment is dS/dt, and this quantity is equal to the rate at which interest accrues, which is the interest rate r times the current value of the investment $S(t)$. Thus

$$dS/dt = rS \tag{11}$$

is the differential equation that governs the process. Suppose that we also know the value of the investment at some particular time, say,

$$S(0) = S_0. \tag{12}$$

Then the solution of the initial value problem (11), (12) gives the balance $S(t)$ in the account at any time t. This initial value problem is readily solved, since the differential equation (11) is both linear and separable. Consequently, by solving Eqs. (11) and (12), we find that

$$S(t) = S_0 e^{rt}. \tag{13}$$

Thus a bank account with continuously compounding interest grows exponentially.

Let us now compare the results from this continuous model with the situation in which compounding occurs at finite time intervals. If interest is compounded once a year, then after t years

$$S(t) = S_0(1 + r)^t.$$

If interest is compounded twice a year, then at the end of 6 months the value of the investment is $S_0[1 + (r/2)]$, and at the end of 1 year it is $S_0[1 + (r/2)]^2$. Thus, after t years we have

$$S(t) = S_0 \left(1 + \frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded m times per year, then

$$S(t) = S_0 \left(1 + \frac{r}{m}\right)^{mt}. \tag{14}$$

The relation between formulas (13) and (14) is clarified if we recall from calculus that

$$\lim_{m \to \infty} S_0 \left(1 + \frac{r}{m}\right)^{mt} = S_0 e^{rt}.$$

The same model applies equally well to more general investments in which dividends and perhaps capital gains can also accumulate, as well as interest. In recognition of this fact, we will from now on refer to r as the rate of return.

Table 2.3.1 shows the effect of changing the frequency of compounding for a return rate r of 8%. The second and third columns are calculated from Eq. (14) for quarterly and daily compounding, respectively, and the fourth column is calculated from Eq. (13) for continuous compounding. The results show that the frequency of compounding is not particularly important in most cases. For example, during a 10-year period the difference between quarterly and continuous compounding is $17.50 per $1000 invested, or less than $2/year. The difference would be somewhat greater for higher rates of return and less for lower rates. From the first row in the table, we see that for the return rate $r = 8\%$, the annual yield for quarterly compounding is 8.24% and for daily or continuous compounding it is 8.33%.

TABLE 2.3.1 Growth of Capital at a Return Rate $r = 8\%$ for Several Modes of Compounding

Years	$S(t)/S(t_0)$ from Eq. (14)		$S(t)/S(t_0)$ from Eq. (13)
	$m = 4$	$m = 365$	
1	1.0824	1.0833	1.0833
2	1.1717	1.1735	1.1735
5	1.4859	1.4918	1.4918
10	2.2080	2.2253	2.2255
20	4.8754	4.9522	4.9530
30	10.7652	11.0203	11.0232
40	23.7699	24.5239	24.5325

Returning now to the case of continuous compounding, let us suppose that there may be deposits or withdrawals in addition to the accrual of interest, dividends, or capital gains. If we assume that the deposits or withdrawals take place at a constant rate k, then Eq. (11) is replaced by

$$dS/dt = rS + k,$$

or, in standard form,

$$dS/dt - rS = k, \tag{15}$$

where k is positive for deposits and negative for withdrawals.

Equation (15) is linear with the integrating factor e^{-rt}, so its general solution is

$$S(t) = ce^{rt} - (k/r),$$

where c is an arbitrary constant. To satisfy the initial condition (12) we must choose $c = S_0 + (k/r)$. Thus the solution of the initial value problem (15), (12) is

$$S(t) = S_0 e^{rt} + (k/r)(e^{rt} - 1). \tag{16}$$

The first term in expression (16) is the part of $S(t)$ that is due to the return accumulated on the initial amount S_0, while the second term is the part that is due to the deposit or withdrawal rate k.

The advantage of stating the problem in this general way without specific values for S_0, r, or k lies in the generality of the resulting formula (16) for $S(t)$. With this formula we can readily compare the results of different investment programs or different rates of return.

For instance, suppose that one opens an individual retirement account (IRA) at age 25 and makes annual investments of $2000 thereafter in a continuous manner. Assuming a rate of return of 8%, what will be the balance in the IRA at age 65? We have $S_0 = 0$, $r = 0.08$, and $k = \$2000$, and we wish to determine $S(40)$. From Eq. (16) we have

$$S(40) = (25,000)(e^{3.2} - 1) = \$588,313. \tag{17}$$

It is interesting to note that the total amount invested is $80,000, so the remaining amount of $508,313 results from the accumulated return on the investment. The balance after 40 years is also fairly sensitive to the assumed rate. For instance, $S(40) = \$508,948$ if $r = 0.075$ and $S(40) = \$681,508$ if $r = 0.085$.

Let us now examine the assumptions that have gone into the model. First, we have assumed that the return is compounded continuously and that additional capital is invested continuously. Neither of these is true in an actual financial situation. We have also assumed that the return rate r is constant for the entire period involved, whereas in fact it is likely to fluctuate considerably. Although we cannot reliably predict future rates, we can use expression (16) to determine the approximate effect of different rate projections. It is also possible to consider r and k in Eq. (15) to be functions of t rather than constants; in that case, of course, the solution may be much more complicated than Eq. (16).

The initial value problem (15), (12) and the solution (16) can also be used to analyze a number of other financial situations, including annuities, mortgages, and automobile loans among others.

EXAMPLE 3

Chemicals in a Pond

Consider a pond that initially contains 10 million gal of fresh water. Water containing an undesirable chemical flows into the pond at the rate of 5 million gal/year and the mixture in the pond flows out at the same rate. The concentration $\gamma(t)$ of chemical in the incoming water varies periodically with time according to the expression $\gamma(t) = 2 + \sin 2t$ g/gal. Construct a mathematical model of this flow process and determine the amount of chemical in the pond at any time. Plot the solution and describe in words the effect of the variation in the incoming concentration.

Since the incoming and outgoing flows of water are the same, the amount of water in the pond remains constant at 10^7 gal. Let us denote time by t, measured in years, and the chemical by $Q(t)$, measured in grams. This example is similar to Example 1 and the same inflow/outflow principle applies. Thus

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out},$$

where "rate in" and "rate out" refer to the rates at which the chemical flows into and out of the pond, respectively. The rate at which the chemical flows in is given by

$$\text{rate in} = (5 \times 10^6) \text{ gal/yr } (2 + \sin 2t) \text{ g/gal}. \tag{18}$$

The concentration of chemical in the pond is $Q(t)/10^7$ g/gal, so the rate of flow out is

$$\text{rate out} = (5 \times 10^6) \text{ gal/yr}[Q(t)/10^7] \text{ g/gal} = Q(t)/2 \text{ g/yr}. \tag{19}$$

Thus we obtain the differential equation

$$\frac{dQ}{dt} = (5 \times 10^6)(2 + \sin 2t) - \frac{Q(t)}{2}, \tag{20}$$

where each term has the units of g/yr.

To make the coefficients more manageable, it is convenient to introduce a new dependent variable defined by $q(t) = Q(t)/10^6$ or $Q(t) = 10^6 q(t)$. This means that $q(t)$ is measured in millions of grams, or megagrams. If we make this substitution in Eq. (20), then each term contains the factor 10^6, which can be cancelled. If we also transpose the term involving $q(t)$ to the left side of the equation, we finally have

$$\frac{dq}{dt} + \tfrac{1}{2}q = 10 + 5\sin 2t. \tag{21}$$

Originally, there is no chemical in the pond, so the initial condition is

$$q(0) = 0. \tag{22}$$

Equation (21) is linear and, although the right side is a function of time, the coefficient of q is a constant. Thus the integrating factor is $e^{t/2}$. Multiplying Eq. (21) by this factor and integrating the resulting equation, we obtain the general solution

$$q(t) = 20 - \tfrac{40}{17}\cos 2t + \tfrac{10}{17}\sin 2t + ce^{-t/2}. \tag{23}$$

The initial condition (22) requires that $c = -300/17$, so the solution of the initial value problem (21), (22) is

$$q(t) = 20 - \tfrac{40}{17}\cos 2t + \tfrac{10}{17}\sin 2t - \tfrac{300}{17}e^{-t/2}. \tag{24}$$

A plot of the solution (24) is shown in Figure 2.3.3, along with the line $q = 20$. The exponential term in the solution is important for small t, but diminishes rapidly as t increases. Later, the solution consists of an oscillation, due to the $\sin 2t$ and $\cos 2t$ terms, about the constant level $q = 20$. Note that if the $\sin 2t$ term were not present in Eq. (21), then $q = 20$ would be the equilibrium solution of that equation.
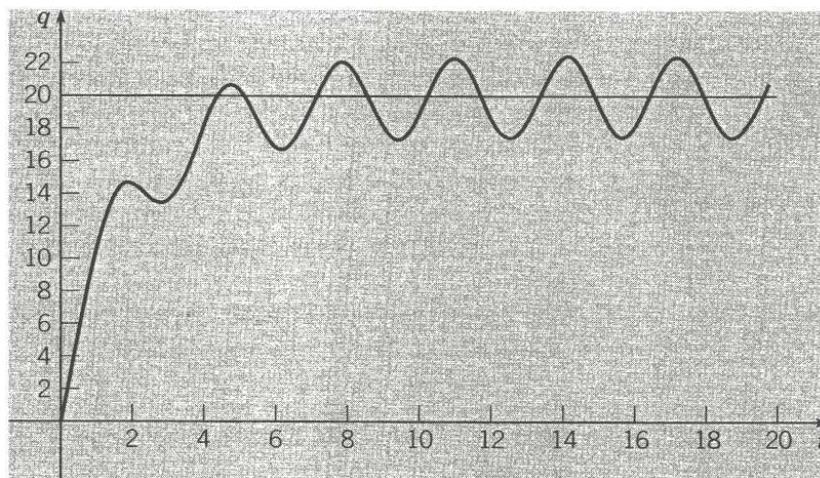

**FIGURE 2.3.3** Solution of the initial value problem (21), (22).

Let us now consider the adequacy of the mathematical model itself for this problem. The model rests on several assumptions that have not yet been stated explicitly. In the first place, the amount of water in the pond is controlled entirely by the rates of flow in and out—none is lost by evaporation or by seepage into the ground, or gained by rainfall. Further, the same is also true of the chemical; it flows in and out of the pond, but none is absorbed by fish or other organisms living in the pond. In addition, we assume that the concentration of chemical in the pond is uniform throughout the pond. Whether the results obtained from the model are accurate depends strongly on the validity of these simplifying assumptions.

A body of constant mass m is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity v_0. Assuming that there is no air resistance, but taking into account the variation of the earth's gravitational field with distance, find an expression for the velocity during the ensuing motion. Also find the initial velocity that is required to lift the body to a given maximum altitude ξ above the surface of the earth, and the smallest initial velocity for which the body will not return to the earth; the latter is the **escape velocity**.

Let the positive x-axis point away from the center of the earth along the line of motion with $x = 0$ lying on the earth's surface; see Figure 2.3.4. The figure is drawn
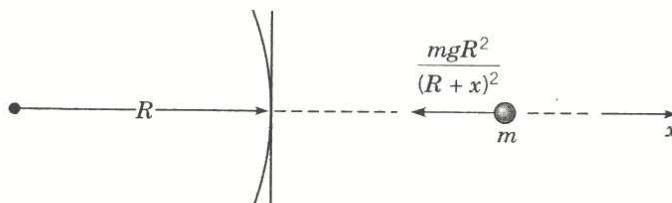
FIGURE 2.3.4   A body in the earth's gravitational field.

horizontally to remind you that gravity is directed toward the center of the earth, which is not necessarily downward from a perspective away from the earth's surface. The gravitational force acting on the body (that is, its weight) is inversely proportional to the square of the distance from the center of the earth and is given by $w(x) = -k/(x + R)^2$, where k is a constant, R is the radius of the earth, and the minus sign signifies that $w(x)$ is directed in the negative x direction. We know that on the earth's surface $w(0)$ is given by $-mg$, where g is the acceleration due to gravity at sea level. Therefore $k = mgR^2$ and

$$w(x) = -\frac{mgR^2}{(R + x)^2}. \tag{25}$$

Since there are no other forces acting on the body, the equation of motion is

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R + x)^2}, \tag{26}$$

and the initial condition is

$$v(0) = v_0. \tag{27}$$

Unfortunately, Eq. (26) involves too many variables since it depends on t, x, and v. To remedy this situation we can eliminate t from Eq. (26) by thinking of x, rather than t, as the independent variable. Then we must express dv/dt in terms of dv/dx by the chain rule; hence

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

and Eq. (26) is replaced by

$$v\frac{dv}{dx} = -\frac{gR^2}{(R + x)^2}. \tag{28}$$

Equation (28) is separable but not linear, so by separating the variables and integrating we obtain

$$\frac{v^2}{2} = \frac{g R^2}{R+x} + c. \tag{29}$$

Since $x = 0$ when $t = 0$, the initial condition (27) at $t = 0$ can be replaced by the condition that $v = v_0$ when $x = 0$. Hence $c = (v_0^2/2) - gR$ and

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}. \tag{30}$$

Note that Eq. (30) gives the velocity as a function of altitude rather than as a function of time. The plus sign must be chosen if the body is rising, and the minus sign if it is falling back to earth.

To determine the maximum altitude ξ that the body reaches, we set $v = 0$ and $x = \xi$ in Eq. (30) and then solve for ξ, obtaining

$$\xi = \frac{v_0^2 R}{2gR - v_0^2}. \tag{31}$$

Solving Eq. (31) for v_0, we find the initial velocity required to lift the body to the altitude ξ, namely,

$$v_0 = \sqrt{2gR\frac{\xi}{R+\xi}}. \tag{32}$$

The escape velocity v_e is then found by letting $\xi \to \infty$. Consequently,

$$v_e = \sqrt{2gR}. \tag{33}$$

The numerical value of v_e is approximately 6.9 miles/sec or 11.1 km/sec.

The preceding calculation of the escape velocity neglects the effect of air resistance, so the actual escape velocity (including the effect of air resistance) is somewhat higher. On the other hand, the effective escape velocity can be significantly reduced if the body is transported a considerable distance above sea level before being launched. Both gravitational and frictional forces are thereby reduced; air resistance, in particular, diminishes quite rapidly with increasing altitude. You should keep in mind also that it may well be impractical to impart too large an initial velocity instantaneously; space vehicles, for instance, receive their initial acceleration during a period of a few minutes.

PROBLEMS

1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 liters of a dye solution with a concentration of 1 g/liter. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 liters/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

2. A tank initially contains 120 liters of pure water. A mixture containing a concentration of γ g/liter of salt enters the tank at a rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

3. A tank originally contains 100 gal of fresh water. Then water containing $\frac{1}{2}$ lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

4. A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

▶ 5. A tank contains 100 gallons of water and 50 oz of salt. Water containing a salt concentration of $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.
   (a) Find the amount of salt in the tank at any time.
   (b) Plot the solution for a time period long enough so that you see the ultimate behavior of the graph.
   (c) The long-time behavior of the solution is an oscillation about a certain constant level. What is this level? What is the amplitude of the oscillation?

6. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.
   (a) Find the time T required for the original sum to double in value as a function of r.
   (b) Determine T if $r = 7\%$.
   (c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

7. A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.
   (a) Determine the sum $S(t)$ accumulated at any time t.
   (b) If $r = 7.5\%$, determine k so that $1 million will be available for retirement in 40 years.
   (c) If $k = \$2000$/year, determine the return rate r that must be obtained to have $1 million available in 40 years.

▶ 8. Person A opens an IRA at age 25, contributes $2000/year for 10 years, but makes no additional contributions thereafter. Person B waits until age 35 to open an IRA and contributes $2000/year for 30 years. There is no initial investment in either case.
   (a) Assuming a return rate of 8%, what is the balance in each IRA at age 65?
   (b) For a constant, but unspecified, return rate r, determine the balance in each IRA at age 65 as a function of r.
   (c) Plot the difference in the balances from part (b) for $0 \leq r \leq 0.10$.
   (d) Determine the return rate for which the two IRA's have equal value at age 65.

9. A certain college graduate borrows $8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate k, determine the payment rate k that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

10. A home buyer can afford to spend no more than $800/month on mortgage payments. Suppose that the interest rate is 9% and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.
    (a) Determine the maximum amount that this buyer can afford to borrow.
    (b) Determine the total interest paid during the term of the mortgage.

11. How are the answers to Problem 10 changed if the term of the mortgage is 30 years?

▶ 12. A recent college graduate borrows \$100,000 at an interest rate of 9% to purchase a con-
dominium. Anticipating steady salary increases, the buyer expects to make payments at a
monthly rate of $800(1 + t/120)$, where t is the number of months since the loan was made.
(a) Assuming that this payment schedule can be maintained, when will the loan be fully
paid?
(b) Assuming the same payment schedule, how large a loan could be paid off in exactly
20 years?

13. A retired person has a sum $S(t)$ invested so as to draw interest at an annual rate r com-
pounded continuously. Withdrawals for living expenses are made at a rate of k dollars/year;
assume that the withdrawals are made continuously.
(a) If the initial value of the investment is S_0, determine $S(t)$ at any time.
(b) Assuming that S_0 and r are fixed, determine the withdrawal rate k_0 at which $S(t)$ will
remain constant.
(c) If k exceeds the value k_0 found in part (b), then $S(t)$ will decrease and ultimately
become zero. Find the time T at which $S(t) = 0$.
(d) Determine T if $r = 8\%$ and $k = 2k_0$.
(e) Suppose that a person retiring with capital S_0 wishes to withdraw funds at an annual
rate k for not more than T years. Determine the maximum possible rate of withdrawal.
(f) How large an initial investment is required to permit an annual withdrawal of \$12,000
for 20 years, assuming an interest rate of 8%?

14. **Radiocarbon Dating.**   An important tool in archeological research is radiocarbon dating.
This is a means of determining the age of certain wood and plant remains, hence of animal
or human bones or artifacts found buried at the same levels. The procedure was developed
by the American chemist Willard Libby (1908–1980) in the early 1950s and resulted in his
winning the Nobel prize for chemistry in 1960. Radiocarbon dating is based on the fact that
some wood or plant remains contain residual amounts of carbon-14, a radioactive isotope of
carbon. This isotope is accumulated during the lifetime of the plant and begins to decay at
its death. Since the half-life of carbon-14 is long (approximately 5730 years[1]), measurable
amounts of carbon-14 remain after many thousands of years. Libby showed that if even
a tiny fraction of the original amount of carbon-14 is still present, then by appropriate
laboratory measurements the *proportion* of the original amount of carbon-14 that remains
can be accurately determined. In other words, if $Q(t)$ is the amount of carbon-14 at time t
and Q_0 is the original amount, then the ratio $Q(t)/Q_0$ can be determined, at least if this
quantity is not too small. Present measurement techniques permit the use of this method
for time periods up to about 50,000 years, after which the amount of carbon-14 remaining
is only about 0.00236 of the original amount.
(a) Assuming that Q satisfies the differential equation $Q' = -rQ$, determine the decay
constant r for carbon-14.
(b) Find an expression for $Q(t)$ at any time t, if $Q(0) = Q_0$.
(c) Suppose that certain remains are discovered in which the current residual amount of
carbon-14 is 20% of the original amount. Determine the age of these remains.

15. The population of mosquitoes in a certain area increases at a rate proportional to the
current population and, in the absence of other factors, the population doubles each week.
There are 200,000 mosquitoes in the area initially, and predators (birds, etc.) eat 20,000
mosquitoes/day. Determine the population of mosquitoes in the area at any time.

▶ 16. Suppose that a certain population has a growth rate that varies with time and that this
population satisfies the differential equation

$$dy/dt = (0.5 + \sin t)y/5.$$

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[1]*McGraw-Hill Encyclopedia of Science and Technology* (8th ed.) (New York: McGraw-Hill, 1997), Vol. 5, p. 48.

(a) If $y(0) = 1$, find (or estimate) the time τ at which the population has doubled. Choose other initial conditions and determine whether the doubling time τ depends on the initial population.

(b) Suppose that the growth rate is replaced by its average value $1/10$. Determine the doubling time τ in this case.

(c) Suppose that the term $\sin t$ in the differential equation is replaced by $\sin 2\pi t$; that is, the variation in the growth rate has a substantially higher frequency. What effect does this have on the doubling time τ?

(d) Plot the solutions obtained in parts (a), (b), and (c) on a single set of axes.

▶ 17. Suppose that a certain population satisfies the initial value problem

$$dy/dt = r(t)y - k, \qquad y(0) = y_0,$$

where the growth rate $r(t)$ is given by $r(t) = (1 + \sin t)/5$ and k represents the rate of predation.

(a) Suppose that $k = 1/5$. Plot y versus t for several values of y_0 between $1/2$ and 1.

(b) Estimate the critical initial population y_c below which the population will become extinct.

(c) Choose other values of k and find the corresponding y_c for each one.

(d) Use the data you have found in parts (a) and (b) to plot y_c versus k.

18. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of $200°F$ when freshly poured, and 1 min later has cooled to $190°F$ in a room at $70°F$, determine when the coffee reaches a temperature of $150°F$.

19. Suppose that a room containing 1200 ft^3 of air is originally free of carbon monoxide. Beginning at time $t = 0$ cigarette smoke, containing 4% carbon monoxide, is introduced into the room at a rate of 0.1 ft^3/min, and the well-circulated mixture is allowed to leave the room at the same rate.

(a) Find an expression for the concentration $x(t)$ of carbon monoxide in the room at any time $t > 0$.

(b) Extended exposure to a carbon monoxide concentration as low as 0.00012 is harmful to the human body. Find the time τ at which this concentration is reached.

20. Consider a lake of constant volume V containing at time t an amount $Q(t)$ of pollutant, evenly distributed throughout the lake with a concentration $c(t)$, where $c(t) = Q(t)/V$. Assume that water containing a concentration k of pollutant enters the lake at a rate r, and that water leaves the lake at the same rate. Suppose that pollutants are also added directly to the lake at a constant rate P. Note that the given assumptions neglect a number of factors that may, in some cases, be important; for example, the water added or lost by precipitation, absorption, and evaporation; the stratifying effect of temperature differences in a deep lake; the tendency of irregularities in the coastline to produce sheltered bays; and the fact that pollutants are not deposited evenly throughout the lake, but (usually) at isolated points around its periphery. The results below must be interpreted in the light of the neglect of such factors as these.

(a) If at time $t = 0$ the concentration of pollutant is c_0, find an expression for the concentration $c(t)$ at any time. What is the limiting concentration as $t \to \infty$?

(b) If the addition of pollutants to the lake is terminated ($k = 0$ and $P = 0$ for $t > 0$), determine the time interval T that must elapse before the concentration of pollutants is reduced to 50% of its original value; to 10% of its original value.

(c) Table 2.3.2 contains data[2] for several of the Great Lakes. Using these data determine

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[2]This problem is based on R. H. Rainey, "Natural Displacement of Pollution from the Great Lakes," *Science 155* (1967), pp. 1242–1243; the information in the table was taken from that source.

TABLE 2.3.2 Volume and Flow Data for the Great Lakes

Lake	$V (\text{km}^3 \times 10^3)$	$r (\text{km}^3/\text{year})$
Superior	12.2	65.2
Michigan	4.9	158
Erie	0.46	175
Ontario	1.6	209

from part (b) the time T necessary to reduce the contamination of each of these lakes to 10% of the original value.

▶ 21. A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/sec from the roof of a building 30 m high. Neglect air resistance.
(a) Find the maximum height above the ground that the ball reaches.
(b) Assuming that the ball misses the building on the way down, find the time that it hits the ground.
(c) Plot the graphs of velocity and position versus time.

▶ 22. Assume that the conditions are as in Problem 21 except that there is a force due to air resistance of $|v|/30$, where the velocity v is measured in m/sec.
(a) Find the maximum height above the ground that the ball reaches.
(b) Find the time that the ball hits the ground.
(c) Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problem 21.

▶ 23. Assume that the conditions are as in Problem 21 except that there is a force due to air resistance of $v^2/1325$, where the velocity v is measured in m/sec.
(a) Find the maximum height above the ground that the ball reaches.
(b) Find the time that the ball hits the ground.
(c) Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problems 21 and 22.

▶ 24. A sky diver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5000 ft, and opens the parachute after 10 sec of free fall. Assume that the force of air resistance is $0.75|v|$ when the parachute is closed and $12|v|$ when the parachute is open, where the velocity v is measured in ft/sec.
(a) Find the speed of the sky diver when the parachute opens.
(b) Find the distance fallen before the parachute opens.
(c) What is the limiting velocity v_L after the parachute opens?
(d) Determine how long the sky diver is in the air after the parachute opens.
(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

25. A body of constant mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Neglect changes in the gravitational force.
(a) Find the maximum height x_m attained by the body and the time t_m at which this maximum height is reached.
(b) Show that if $kv_0/mg < 1$, then t_m and x_m can be expressed as

$$t_m = \frac{v_0}{g}\left[1 - \frac{1}{2}\frac{kv_0}{mg} + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right],$$

$$x_m = \frac{v_0^2}{2g}\left[1 - \frac{2}{3}\frac{kv_0}{mg} + \frac{1}{2}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right].$$

(c) Show that the quantity kv_0/mg is dimensionless.

26. A body of mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Assume that the gravitational attraction of the earth is constant.

(a) Find the velocity $v(t)$ of the body at any time.

(b) Use the result of part (a) to calculate the limit of $v(t)$ as $k \to 0$, that is, as the resistance approaches zero. Does this result agree with the velocity of a mass m projected upward with an initial velocity v_0 in a vacuum?

(c) Use the result of part (a) to calculate the limit of $v(t)$ as $m \to 0$, that is, as the mass approaches zero.

27. A body falling in a relatively dense fluid, oil for example, is acted on by three forces (see Figure 2.3.5): a resistive force R, a buoyant force B, and its weight w due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes'[3] law $R = 6\pi\mu a|v|$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid.
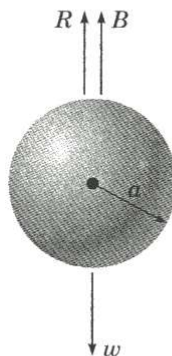

FIGURE 2.3.5   A body falling in a dense fluid.

(a) Find the limiting velocity of a solid sphere of radius a and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ.

(b) In 1910 the American physicist R. A. Millikan (1868–1953) determined the charge on an electron by studying the motion of tiny droplets of oil falling in an electric field. A field of strength E exerts a force Ee on a droplet with charge e. Assume that E has been adjusted so the droplet is held stationary ($v = 0$), and that w and B are as given above. Find a formula for e. Millikan was able to identify e as the charge on an electron and to determine that $e = 4.803 \times 10^{-10}$ esu.

▶ 28. A mass of 0.25 kg is dropped from rest in a medium offering a resistance of $0.2|v|$, where v is measured in m/sec.

(a) If the mass is dropped from a height of 30 m, find its velocity when it hits the ground.

(b) If the mass is to attain a velocity of no more than 10 m/sec, find the maximum height from which it can be dropped.

(c) Suppose that the resistive force is $k|v|$, where v is measured in m/sec and k is a constant. If the mass is dropped from a height of 30 m and must hit the ground with a velocity of no more than 10 m/sec, determine the coefficient of resistance k that is required.

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[3]George Gabriel Stokes (1819–1903), professor at Cambridge, was one of the foremost applied mathematicians of the nineteenth century. The basic equations of fluid mechanics (the Navier–Stokes equations) are named partly in his honor, and one of the fundamental theorems of vector calculus bears his name. He was also one of the pioneers in the use of divergent (asymptotic) series, a subject of great interest and importance today.

29. Suppose that a rocket is launched straight up from the surface of the earth with initial velocity $v_0 = \sqrt{2gR}$, where R is the radius of the earth. Neglect air resistance.
    (a) Find an expression for the velocity v in terms of the distance x from the surface of the earth.
    (b) Find the time required for the rocket to go 240,000 miles (the approximate distance from the earth to the moon). Assume that $R = 4000$ miles.

30. Find the escape velocity for a body projected upward with an initial velocity v_0 from a point $x_0 = \xi R$ above the surface of the earth, where R is the radius of the earth and ξ is a constant. Neglect air resistance. Find the initial altitude from which the body must be launched in order to reduce the escape velocity to 85% of its value at the earth's surface.

▶ 31. Let $v(t)$ and $w(t)$ be the horizontal and vertical components of the velocity of a batted (or thrown) baseball. In the absence of air resistance, v and w satisfy the equations

$$dv/dt = 0, \qquad dw/dt = -g.$$

(a) Show that

$$v = u \cos A, \qquad w = -gt + u \sin A,$$

where u is the initial speed of the ball and A is its initial angle of elevation.
(b) Let $x(t)$ and $y(t)$, respectively, be the horizontal and vertical coordinates of the ball at time t. If $x(0) = 0$ and $y(0) = h$, find $x(t)$ and $y(t)$ at any time t.
(c) Let $g = 32$ ft/sec^2, $u = 125$ ft/sec, and $h = 3$ ft. Plot the trajectory of the ball for several values of the angle A, that is, plot $x(t)$ and $y(t)$ parametrically.
(d) Suppose the outfield wall is at a distance L and has height H. Find a relation between u and A that must be satisfied if the ball is to clear the wall.
(e) Suppose that $L = 350$ ft and $H = 10$ ft. Using the relation in part (d), find (or estimate from a plot) the range of values of A that correspond to an initial velocity of $u = 110$ ft/sec.
(f) For $L = 350$ and $H = 10$ find the minimum initial velocity u and the corresponding optimal angle A for which the ball will clear the wall.

▶ 32. A more realistic model (than that in Problem 31) of a baseball in flight includes the effect of air resistance. In this case the equations of motion are

$$dv/dt = -rv, \qquad dw/dt = -g - rw,$$

where r is the coefficient of resistance.
(a) Determine $v(t)$ and $w(t)$ in terms of initial speed u and initial angle of elevation A.
(b) Find $x(t)$ and $y(t)$ if $x(0) = 0$ and $y(0) = h$.
(c) Plot the trajectory of the ball for $r = 1/5$, $u = 125$, $h = 3$, and for several values of A. How do the trajectories differ from those in Problem 31 with $r = 0$?
(d) Assuming that $r = 1/5$ and $h = 3$, find the minimum initial velocity u and the optimal angle A for which the ball will clear a wall that is 350 ft distant and 10 ft high. Compare this result with that in Problem 31(f).

33. **Brachistochrone Problem.** One of the famous problems in the history of mathematics is the brachistochrone problem: to find the curve along which a particle will slide without friction in the minimum time from one given point P to another Q, the second point being lower than the first but not directly beneath it (see Figure 2.3.6). This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli, and by Isaac Newton, Gottfried Leibniz, and Marquis de L'Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem it is convenient to take the origin as the upper point P and to orient the axes as shown in Figure 2.3.6. The lower point Q has coordinates (x_0, y_0). It is