INSTRUCTIONS

1. Fill in the information requested above.

2. Make sure you have a complete test. There are 7 pages (including this cover page) and 6 problems.

3. You have to show your work in the space provided to receive partial credit.

4. No books and no notes in this test. Calculators are allowed.

5. This is an 120 minute exam.

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1. (20 points) To find the solution $\sqrt{a}$ to the equation $x^2 - a = 0$ ($a > 0$), one has several ways to do the fixed point iteration:

$$x_{n+1} = \varphi(x_n), \quad n = 0, 1, 2, \ldots,$$

with

$$\varphi(x) = \frac{1}{2} \left( x + \frac{a}{x} \right), \quad \varphi(x) = \frac{a}{x}, \quad \varphi(x) = 2x - \frac{a}{x}.$$

Discuss the convergence (or nonconvergence) behavior for each of the above three iteration functions. In the case of convergence, determine the order of convergence.
2. (20 points) Suppose $f$ is a function on the interval $[0,3]$ for which one knows that

$$f(0) = 1, \quad f(1) = 2, \quad f'(1) = -1, \quad f(3) = f'(3) = 0.$$

(a) Find a polynomial $p(x)$ of lowest degree that satisfies the above conditions and evaluate $p(2)$.

(b) Estimate the maximum possible error of the answer given in part (a) if one knows, in addition, that $|f^{(5)}(x)| \leq M$ on $[0,3]$. Express the answer in terms of $M$. 
3. (15 points) Find the best least squares polynomial approximation of degree 2 to the function \( f(x) = e^x \) on the interval \([0, 1]\). (Recall that the first three Legendre polynomials defined on the interval \([-1, 1]\) are \( \phi_0(x) = 1 \), \( \phi_1(x) = x \), \( \phi_2(x) = \frac{1}{2}(3x^2 - 1) \).)
4. (20 points) Determine the abscissae $x_0, x_1$, and weights $a_0, a_1$ in the Gauss quadrature 

$$\int_0^\infty e^{-x}f(x)\,dx \approx a_0f(x_0) + a_1f(x_1).$$
5. (10 points) The explicit trapezoidal method to solve the equation $y' = f(y)$ reads

$$\begin{cases}
  y^{(1)} &= y^n + \Delta t f(y^n); \\
  y^{n+1} &= y^n + \frac{\Delta t}{2} (f(y^n) + f(y^{(1)})).
\end{cases}$$

Show that this method is second order accurate.
6. (15 points) Derive the second order backward differentiation formula (BDF) for solving the equation $y' = f(y)$ (assume that the uniform time step $\Delta t$ is used).