Score:____________

Numerical Analysis Qualifying Exam, August 2021

Note:

1. **No books and no notes in this test.**
2. **You will need a calculator for this test.**
3. **Show necessary intermediate steps of your work. No credit otherwise.**

1. (10 points)
   A calculator will be needed for this problem. Determine how many significant bits will be lost when evaluating the expression $(x - \sin x)$ at $x = 0.1$.

2. (10 points)
   Consider the system

$$\begin{cases} x^2 - y^2 - yz = -2 \\ e^x - e^y + z - 3 = 0 \\ z^2 - xy = -1. \end{cases}$$

   Start with the initial guess $(x, y, z) = (0, 0, 1)$, and perform one step of the Newton's method to find an estimate to the root of this system.

3. (15 points)
   Assume single precision in the following computations, and let ϵ denote the unit roundoff error in single precision. Show that

$$\left| fl\left(\frac{1}{3}\right) - \frac{1}{3} \right| = \frac{1}{6}\epsilon$$

   where $fl(x)$ denotes the floating-point number corresponding to x.

4. (a) (10 points)
   Consider the linear multistep method of the form

$$hf'_{n+1} = \alpha f_{n+1} + \beta f_n + \gamma f_{n-1}$$

   where $f_n = f(nh)$ and h denotes the step size. By employing the order conditions for the linear multistep method, show that $\alpha = \frac{3}{2}$, $\beta = -2$ and $\gamma = \frac{1}{2}$ in order for this to be a second-order method.

   (b) (10 points)
   Let us consider another way to derive the above approximation,

$$f'(x) = \frac{\frac{3}{2}f(x) - 2f(x-h) + \frac{1}{2}f(x-2h)}{h} + E_r(x)$$

where $E_r(x)$ denotes the error term. By considering the polynomial interpolation of f on the grids $\{x,\ x-h,\ x-2h\}$ and the error relation for the interpolation polynimials, derive this approximation for $f'(x)$ and find the expression for the error term $E_r(x)$. We assume that $f(x)$ is continuously differentiable to a sufficient order.

5. (15 points)
Prove that the iteration

$$x_{n+1} = \cos(x_n), \quad n \geqslant 0$$

always converges irrespective of the starting point.

6. (15 points)
Prove the Marsden's identity

$$\sum_{i=-\infty}^{\infty} (t_{i+1}-s)(t_{i+2}-s)\cdots(t_{i+k}-s)B_i^k(x) = (x-s)^k$$

where s is a constant, and $B_i^k(x)$ is the B-spline function of degree k having knots t_i $(i = 0, \pm 1, \pm 2, \dots)$

7. (15 points)
Suppose $f(x)$ has continuous derivatives up to the order 4. Let r denote a simple root of f, and suppose that $f^{(n)}(r)$ is bounded for $n \leqslant 4$. The Halley's method for solving $f(x) = 0$ is given by

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - \frac{1}{2}f(x_n)f''(x_n)}.$$

By treating this as a fixed point iteration, show that this method is locally convergent and that its rate of convergence is at least cubic for computing r.