QUALIFYING EXAMINATION

JANUARY 10, 1994

MATH 519

1. (20 points) Let X_1, X_2, X_3 and X_4 be independent random variables, each uniformly distributed on the interval $(-1, 1)$. Find

   (a) $P(X_1 < X_2 < X_3 < X_4)$

   (b) $P\left(X_1^2 < (X_1 + X_2)^2\right)$

   (c) $P(X_1^2 > X_2^2 + X_3^2)$

2. (20 points) A fair die is rolled repeatedly until it comes up ace. This procedure is repeated 100 times. Find

   (a) the probability that exactly four threes are rolled in exactly 5 of the 100 repetitions;

   (b) the mean and variance of the total number of threes rolled.

3. (20 points) The number of cars arriving at the McDonald's drive-up window in a given day is a Poisson random variable, N, with parameter λ. The numbers of passengers in these cars are independent random variables, X_i, each equally likely to be one, two, three or four. Find the moment generating function of

$$S = \sum_{i=1}^{N} X_i,$$

   the total number of passengers in all the cars.

4. (20 points) Let $X_1, X_2, \ldots$ be independent random variables, each uniform on the interval $(0, 1)$, and let $S_n = X_1 + X_2 + \ldots X_n$.

   (a) Find $\lim_{n\to\infty} P(S_n \le t)$ for all $t > 0$.

   (b) Find $\lim_{n\to\infty} P(S_n/n \le t)$ for all $t > 0$.

   (c) Find $\lim_{n\to\infty} P((S_n - n/2)/\sqrt{n/12} \le t)$ for all $t > 0$.

   (d) Find $\lim_{n\to\infty} P(\prod_{i=1}^{n} X_i^{1/n} \le t)$ for all $t > 0$.

   (e) Find $\lim_{n\to\infty} P(\max(X_1, \ldots, X_n) \le t)$ for all $t > 0$.

5. (20 points) Let $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_n})$ be a permutation of the integers 1 to n. Let $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \dots, \mathbf{y_n})$ be the sequence of *relative ranks* of the x_i's; i.e., $y_i = k$ if x_i is the k-th *smallest* of the first k x_i's.

   For example, if $n = 5$ and $\mathbf{x} = (\mathbf{2}, \mathbf{1}, \mathbf{4}, \mathbf{5}, \mathbf{3})$, then $\mathbf{y} = (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{4}, \mathbf{3})$.

   (a) If $\mathbf{y} = (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{3}, \mathbf{1})$, what is $\mathbf{x}$?

   (b) Give a rule to compute $\mathbf{x}$'s from $\mathbf{y}$'s.

   (c) The result of part (b) shows that, for any fixed n, there is a one-to-one correspondence between the sets of all possible $\mathbf{x}$'s and all possible $\mathbf{y}$'s. Now suppose $\mathbf{X} = (\mathbf{X_1}, \mathbf{X_2}, \dots, \mathbf{X_n})$ is a *random* permutation of the integers 1 to n, and $\mathbf{Y} = (\mathbf{Y_1}, \mathbf{Y_2}, \dots, \mathbf{Y_n})$ are the corresponding relative ranks. Show that the Y_i's are independent, with each Y_i uniformly distributed on the integers 1 to i.