QUALIFYING EXAMINATION

JANUARY 1996

MATH 519

1. Ten balls are thrown randomly into ten boxes. A box can hold any number of balls. Find the mean and variance of the number of empty boxes.

2. A fair six sided die is rolled repeatedly until the first six comes up. Let N_i be the number of times the side with i dots comes up, $i = 1, 2, 3, 4, 5$.
   (a) Find the distribution of N_1.
   (b) Find the distribution of $N_1 + N_2 + N_3 + N_4 + N_5$.
   (c) Are N_1 and $N_1 + N_2 + N_3 + N_4 + N_5$ independent. Why?
   (d) Let Z be the number of those $i = 1, 2, 3, 4, 5$ such that $N_i > 0$. Find the distribution of Z.

3. Let X and Y be independent exponential ($\lambda = 1$) variables, that is, they have probability density function $f(t) = e^{-t}, \ t > 0$. Let Z be independent of (X, Y), and suppose $P(Z = 1) = P(Z = -1) = \dfrac{1}{2}$.
   (a) Find a function $\phi(x, y)$ such that $\phi(X, Y)$ has joint density $g(x, y) = 1$ if $0 < x < 1$ and $0 < y < 1, \ g(x, y) = 0$ elsewhere.
   (b) Find the joint density of (ZX, ZY).
   (c) Find the density of $\dfrac{X}{X + Y}$.

4. Let X and Y be independent and identically distributed random variables each with a continuous density $f(t)$ which is zero if $t \notin [0, 1]$, and not zero if $t \in (0, 1)$.
   (a) Find an integer n such that
   $$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} P\left(|(X, Y) - \left(\frac{1}{2}, \frac{1}{2}\right) | \right) = \delta,$$
   where $\delta \in (0, \infty)$ and $|(a, b) - (c, d)|$ is the Euclidean distance between these points. Evaluate δ in terms of f.
   (b) Find an integer n such that
   $$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} P(|X - Y| < \varepsilon) = \delta,$$
   where $\delta \in (0, \infty)$. For which f is this δ minimized?