Mathematics 519
Qualifying Exam
August 1997

1. A standard deck of playing cards is divided into two stacks, one consisting of the 26 black cards (clubs ♣ and spades ♠), and the other consisting of the 26 red cards (diamonds ♢ and hearts ♡). Each stack is thoroughly shuffled. You are then dealt 5 cards, two from the black stack, and three from the red stack. What is the probability that you are dealt at least one Ace?

2. Let $Z, X_1, X_2, \ldots, X_5$ be random variables such that (a) Z has the uniform distribution on the unit interval $(0,1)$; and (b) for any $z \in (0,1)$, conditional on $Z = z$ the random variables $X_1, X_2, \ldots, X_5$ are independent, identically distributed Bernoulli-z, i.e., for any sequence $e_1, e_2, \ldots, e_n$ of zeros and ones,

$$P(X_i = e_i \; \forall \, 1 \leq i \leq n \mid Z = z) = \prod_{i=1}^{5} z^{e_i}(1-z)^{1-e_i}.$$

Find $P\{\sum_{i=1}^{5} X_i = 4\}$.

3. A standard deck of playing cards is thoroughly shuffled. Cards are then dealt face up from the top of the deck, one at a time, until the first Ace appears. Let Y be the number of cards dealt. Calculate EY.

4. Suppose that the random variables $Y, X_1, X_2, \ldots$ are independent, and that

$$P\{Y = n\} = 2^{-n} \qquad \forall \, n = 1, 2, \ldots$$
$$P\{X_k \geq t\} = e^{-\pi t} \qquad \forall \, t > 0 \text{ and } k = 1, 2, \ldots.$$

Let $S_n = X_1 + X_2 + \cdots + X_n$. Calculate $E(S_Y^3)$.

5. Let $\Theta_1, \Theta_2, \ldots$ be a sequence of independent, identically distributed random variables with the uniform distribution on the interval $(0, 2\pi)$. For $n = 1, 2, \ldots$ define

$$X_n = \sum_{k=1}^{n} \cos \Theta_k, \; Y_n = \sum_{k=1}^{n} \sin \Theta_k, \text{ and } R_n^2 = X_n^2 + Y_n^2.$$

Prove that

$$\lim_{n \to \infty} P\{R_n^2 \geq n\}$$

exists, and, if possible, evaluate it.