# QUALIFYING EXAMINATION

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MATH 519 - Professor Ma

1. A deck (deck #1) of cards has $a$ red cards and $b$ black cards, another deck (deck #2) has $\alpha$ red cards and $\beta$ black cards. Both decks are well-shuffled. Suppose you pick $c$ ($c \le a+b$) cards randomly from deck #1 and mix them into deck #2. What is the probability of picking a red card from deck #2 now?

2. A clerk in a gas station is rolling a fair dice while waiting for the customers to come. Suppose that the number of times the dice is rolled between two customers has a Poisson distribution with parameter $\lambda = 5$. Let $\xi$ be the total points (of the dice) the clerk observed right before the next customer comes in, determine $E\xi$ and $D\xi$ (standard deviation).

3. Let $\xi$ and $\eta$ be two random variables, both taking only two values. Show that if they are uncorrelated, then they are independent.

4. Suppose that $\{X_i\}_{i=1}^{\infty}$ is an i.i.d. sequence with density function $p(x)$; and $\{A_\varepsilon\}_{\varepsilon>0}$ is a family of events such that

$$0 < \gamma_\varepsilon := \int_{A_\varepsilon} p(x)dx = P\{X_1 \in A_\varepsilon\} \to 0, \qquad \text{as } \varepsilon \to 0.$$

   Define, for each $N$, $\hat{\gamma}_N^\varepsilon = \frac{1}{N}\sum_{i=1}^N 1_{\{X_i \in A_\varepsilon\}}$ ($1_B$ is the *indicator function* of set $B$).

   (i) Find $E(\hat{\gamma}_N^\varepsilon)$ and $\mathrm{Var}(\hat{\gamma}_N^\varepsilon)$;

   (ii) Suppose $N$ is large enough. For each $\varepsilon > 0$ and $N > 0$, using the attached Normal Table to determine $z_N^\varepsilon$ such that the probability that $\hat{\gamma}_N^\varepsilon \in (\gamma_\varepsilon - z_N^\varepsilon, \gamma_\varepsilon + z_N^\varepsilon)$ is (approximately) 0.99.

   (iii) Show that $\lim\limits_{\varepsilon\to 0} \dfrac{z_N^\varepsilon}{\hat{\gamma}_N^\varepsilon} = \infty$, a.s., no matter how large $N$ is.

5. Let $\xi$ be a random variable with positive density function $p(x)$. Suppose that $p$ is twice differentiable and satisfies the identity

$$\frac{p'(x+y)}{p(x+y)} + \frac{p'(x-y)}{p(x-y)} = 2\frac{p'(x)}{p(x)}, \qquad \forall x, y \in (-\infty, \infty).$$

   Show that $\xi$ must be a normal random variable.