QUALIFYING EXAMINATION
AUGUST 2000
MATH 519 - Prof. Studden

All problems have the same point value.

1. The annual number of accidents for an individual driver has a Poisson distribution with mean λ. The Poisson means λ, of a heterogeneous population of drivers, have a gamma distribution with mean 0.1 and variance 0.01. Calculate the probability that a driver, selected at random from the population, will have 2 or more accidents in one year. The gamma density is given by

$$f(x) = \frac{1}{\theta}(\frac{x}{\theta})^{\alpha-1}\frac{e^{-\frac{x}{\theta}}}{\Gamma(\alpha)}.$$

2. Let X_n be any sequence of random variable such that $\mathrm{Var}(X_n) \leq c\mu_n$ for some fixed constant c and $\mu_n = EX_n \longrightarrow \infty$. Show that $\lim_{n\longrightarrow\infty} P(X_n > a) = 1$ for all a.

3. For any random variable X and Y determine whether the following are true or false;
a) X and Y - E(Y|X) are uncorrelated,
b) Var(Y - E(Y|X)) = E(Var(Y|X)),
c) Cov(X,E(Y|X)) = Cov(X,Y).

4. An urn contains W white and B black balls. Balls are randomly selected <u>without</u> replacement from the urn until w white balls have been removed. ($1 \leq w \leq W$).
a) If X is the number of black balls that are selected, what is P(X=k)? ($k = 0, 1, \cdots, B$).
b) What is E(X)?

5. Let $S_n = X_1 + \cdots + X_n$ where the X_i are independent and uniformly distributed on (0,1).

   a) What is the moment generating function of S_n?

   b) Show that $f_n(x) = F_{n-1}(x) - F_{n-1}(x-1)$ where f_k and F_k are the density and distribution function of S_k respectively.

   c) Show by induction that

$$f_n(x) = \frac{1}{(n-1)!}\sum_{k=0}^{n}(-1)^k \binom{n}{k} (x-k)_+^{n-1}.$$

   d) Obtain the moment generating function of S_n directly from the density in part c.

6. Let $X_1, \cdots, X_n$ be independent random variables with common distribution which is uniform on the interval (-1/2, 1/2). Show that the random variables

$$Z_n = \sqrt{n}\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^2}$$

converge in distribution to some random variable Z and identify the distribution of Z.

7. Let X_1, X_2, X_3 be independent normal random variables with mean zero and variance one. What is the distribution of

$$\frac{X_1 + X_2 X_3}{\sqrt{1 + X_3^2}}?$$