MA 519 Introduction to Probability
January 2000, Qualifying Examination

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- This qualifying exam contains **five** questions.
- By all means **simplify** your answers as much as possible.
- It might be useful to know that for any positive integers m and n,
$$\int_0^1 x^m(1-x)^n\,dx = \frac{m!\,n!}{(m+n+1)!}$$
- A normal table is provided at the end.

1. There are n people among whom are A and B. They stand in a row randomly. What is the probability that there will be exactly r people between A and B?

   What is the corresponding probability if they stand in a circular ring? (In this case, consider only the arc going from A to B in the positive (i.e. counter-clockwise) direction.)

2. Consider a large collection of coins such that the probability p of a coin giving a head is itself a random variable which is uniformly distributed in $[0,1]$.

   Let X be the total number of heads in n tossing of the coins. Find $P(X=i)$ $(i = 0,1,\ldots n)$ in the following two situations:

   (a) Pick a coin at random and then toss *this* coin n times.

   (b) Pick a coin at random for *each* tossing.

3. Consider a sequence of Bernoulli trials of tossing a coin with p as the probability of giving a head. Let X be the number of trials for the m-th head to occur. Find the moment generating function $M_X(s)$ of X.

   (Note: Given any positive random variable X, its moment generating function $M_X(s)$ is defined as $E(e^{-sX})$.)

4. Let X and Y be two *independent*, identically and exponentially distributed random variables:
$$\begin{aligned} P(X\in(x,x+dx)) &= \lambda e^{-\lambda x}dx, \quad x\geq 0 \\ P(Y\in(y,y+dy)) &= \lambda e^{-\lambda y}dy, \quad y\geq 0 \end{aligned}$$

   Let $T_1 = \min(X,Y)$, $T_2 = \max(X,Y)$ and $W = T_2 - T_1$.

(a) Find the probability density functions of T_1, T_2 and W.

(b) Find the joint probability density function of T_1 and W.

(c) Are T_1 and W independent?

5. There are 100 light bulbs whose lifetimes T's are independent exponentials with mean 5 hours (i.e. the probability density function of T is $\frac{1}{5}e^{-\frac{1}{5}t}$ for $t \geq 0$). If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one. In addition, it takes a random time, uniformly distributed over $(0, 0.5)$ to replace a failed bulb. Approximate the probability that all bulbs have failed by time 550?