PROBABILITY QUALIFYING EXAMINATION

August 2012

1. A particle performing a three dimensional random walk starts at time $n = 0$ at the origin $(0, 0, 0)$. At each subsequent time $n = 1, 2, 3, \ldots$, the particle moves exactly 1 unit in one direction: either right, left, forward, backward, up, or down. Each of these six possible moves occurs with an equal probability of $1/6$. The particle makes independent movement decisions at different times. Calculate the probability that, at time $n = 6$, the particle is at the origin.

2. Suppose X, Y, Z are three iid standard normal random variables. Identify with proof a nonnegative continuous function g such that $\frac{X+YZ}{g(Z)} \sim N(0, 1)$.

3. Explicitly evaluate with proof

$$\lim_{n\to\infty} e^{-n}\left[1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right].$$

4. Suppose m balls are distributed in a completely random and mutually independent way into n bins. Let $W_{m,n}$ denote the number of bins that remain empty.

a. For each fixed m, n, find the mean and variance of $W_{m,n}$.

b. Suppose there exists a fixed λ with $0 < \lambda < 1$ such that $m/n \to \lambda$ as $n \to \infty$. Show that there exists a constant $c(\lambda)$ such that

$$\frac{W_{m,n}}{n} \xrightarrow{P} c(\lambda),$$

and identify the value $c(\lambda)$.

5. Suppose X and Y have joint density

$$f_{X,Y}(x, y) = \begin{cases} 1/x^3 & \text{if } x > 1 \text{ and } 0 < y < x, \\ 0 & \text{otherwise.} \end{cases}$$

Find the density of $X - Y$.

6. Assume that X is a positive random variable such that $\lim_{x\to+\infty} \exp(\sqrt{x})\,\mathbf{P}(X > x) = L > 0$ for some finite L. Compute $\sup\{p > 0 : \mathbf{E}[\exp(X^p)] < \infty\}$.

7. Let $\{X_n\}_{n\geq 0}$ be an irreducible, stationary, reversible Markov chain. Prove that, if the chain is periodic, then the period can only be a unique integer m. Identify with proof the value of m.

8. Alpha particles and beta particles are the result of radioactive decay. In a lab experiment, there is exactly one α-particle counter, and there are an infinite number of β-particle counters. All counters count independently—according to a Poisson process at rate 1—once they are turned on. At time 0, the α-particle counter is turned on, and all the β-particle counters are turned off. Each time an α particle is counted, exactly 1 new β-particle counter is turned on. Once it is on, a counter stays on. Let Y be the number of β particles counted in the interval $[1, 2]$. Find the mean and variance of Y.

9. Let X_t (for $t \geq 0$) denote a pure jump process on the integers, with initial value $X_0 = 0$. When a jump occurs, it is to a nearest neighbor, i.e., a change of ± 1. The generator is the matrix $p'_{i,i+1}(0) = 1$ and $p'_{i,i-1}(0) = 1$, that is, the rate at which a process jumps to a particular neighbor is 1. Prove that $\lim_{t\to+\infty} P(X_t = 0) = 0$.