Probability Qualifier, Mathematics Department, August 2014 , DasGupta

Notes:

Please show detailed work, and write legibly. Answer four of the five problems in Group A and one of the two problems in Group B.

Group A

1. a) 3 balls are distributed one by one and at random in 3 boxes. What is the probability that exactly one box remains empty?
b) n balls are distributed one by one and at random in n boxes. Find the probability that exactly one box remains empty.
c) n balls are distributed one by one and at random in n boxes. Find the probability that exactly two boxes remain empty.
4 + 8 + 8 = 20 points

2. n players each roll a fair die. For any pair of players $i, j, i < j$, who roll the same number, the group is awarded one point.
a) Find the mean of the total points of the group.
b) Find the variance of the total points of the group.
6 + 14 = 20 points

3. Suppose $X_1, X_2, \cdots$ is an infinite sequence of iid $U[0,1]$ variables. Find the limit

$$\lim_{n\to\infty} P\Big(\frac{(\prod_{i=1}^n X_i)^{\frac{1}{n}}}{\frac{\sum_{i=1}^n X_i}{n}} > \frac{3}{4}\Big)$$

20 points

4. Suppose X is an exponential random variable with density $\frac{1}{\sigma_1}e^{-x/\sigma_1}$ and Y is another exponential random variable with density $\frac{1}{\sigma_2}e^{-y/\sigma_2}$, and that X, Y are independent.
a) Find the CDF of $\frac{X}{X+Y}$.
b) In the case $\sigma_1 = 2, \sigma_2 = 1$, find the mean of $\frac{X}{X+Y}$.
12 + 8 = 20 points

5. Ten independently picked $U[0,100]$ numbers are each rounded to the near-

est integer. Use the central limit theorem to approximate the probability that the sum of the ten rounded numbers equals the rounded value of the sum of the ten original numbers.
20 points

Group B

6. Suppose for some given $m \geq 2$, we choose m iid $U[0,1]$ variables $X_1, X_2, \cdots, X_m$.
Let $X_{(1)}$ denote their minimum and $X_{(m)}$ their maximum.
Now continue sampling $X_{m+1}, X_{m+2}, \cdots$ from the $U[0,1]$ density. Let N be the first index k such that X_{m+k} falls outside of the interval $[X_{(1)}, X_{(m)}]$.
a) Find a formula for $P(N > n)$ for a general n.
b) Hence, explicitly find $E(N)$.
10 + 10 = 20 points

7. A $G_{n,p}$ graph on n vertices is obtained by adding each of the $\binom{n}{2}$ possible edges into the graph mutually independently with probability p.
If vertex subsets A, B both have k vertices, and each vertex in A shares an edge with each vertex in B, but there are no edges among the vertices within A or within B, then A, B generate a complete bipartite subgraph of order k denoted as $K_{k,k}$.
a) For given n and p, find an expression for the expected number of complete bipartite subgraphs $K_{3,3}$ of order $k = 3$ in a $G_{n,p}$ graph.
b) Let p_n denote the value of p for which the expected value in part a) equals one. Identify constants α, β such that $\lim_{n\to\infty} n^\alpha p_n = \beta$.
10 + 10 = 20 points