MA 519: Introduction to Probability Theory
August 2015, Qualifying Examination (Yip)

Your PUID: ________________________

This examination contains six questions, totaling 120 points. In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

1. There are two types ($i = 1, 2$) of batteries in a bin. The life span of type i is an exponential random variable which parameter λ_i. The probability of type i battery to be chosen is p_i ($p_1 + p_2 = 1$).

   Suppose a randomly chosen battery is operating at time t hours, what is the probability that it will still be operating after an additional s hours?

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2. Suppose n balls are distributed in n boxes in such a way that each ball chooses a box independently of each other.

   (a) What is the probability that Box #1 is empty?

   (b) What is the probability that only Box #1 is empty?

   (c) What is the probability that only one box is empty?

   (d) Given that Box #1 is empty, what is the probability that only one box is empty?

   (e) Given that only one box is empty, what is the probability that Box #1 is empty?

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3. Let X and Y be two positive continuous random variables having joint density $f(x, y)$.

   (a) Let $W = \frac{Y}{X}$ and $Z = X + Y$. Find the joint density of W and Z in terms of f.

   (b) Now suppose X and Y are two independent Gamma random variables with densities $\Gamma(\alpha_1, \lambda)$ and $\Gamma(\alpha_2, \lambda)$ respectively. Are W and Z (defined in (a)) independent?

   (Note: The density p of $\Gamma(\alpha, \lambda)$ is given by $p(t) = \dfrac{\lambda^\alpha t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)}$.)

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4. Let $X_1, X_2, \ldots X_n$ be independent, identically distributed random variables having mean 0 and variance σ^2. Let $S_n = X_1 + X_2 + \cdots X_n$.

   (a) Suppose X_1 has finite third moment. Show that $E(S_n^3) = nE(X_1^3)$ and hence find the value of

   $$\lim_{n\to\infty} E\left[\left(\frac{S_n}{\sigma\sqrt{n}}\right)^3\right].$$

   (b) Suppose X_1 has finite fourth moment. Show that $E(S_n^4) = nE(X_1^4) + 3n(n-1)\sigma^4$ and hence find the value of

   $$\lim_{n\to\infty} E\left[\left(\frac{S_n}{\sigma\sqrt{n}}\right)^4\right].$$

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5. Let $X_1, X_2, \dots X_n$ be n independent random variables with uniform distribution over $[0, 1]$. Let $Y = \min\{X_1, X_2, \dots X_n\}$ and $Z = \max\{X_1, X_2, \dots X_n\}$.

   (a) Find the pdf of Y.

   (b) Find the pdf of Z.

   (c) Find the joint pdf of Y and Z.

   (d) Find $\mathrm{E}(Z - Y)$.

   (e) Find $\mathrm{Var}(Z - Y)$.

   (A useful integration identity: $\displaystyle\int_0^1 x^m (1-x)^n \, dx = \frac{m!\, n!}{(m+n+1)!}$.)

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6. Let X be a standard normal Gaussian random variable (the mean zero and variance one). Let $a, b > 0$ such that $a - b > 0$ and $\epsilon > 0$. Prove the following two statements:

   (a) $\lim_{\epsilon \longrightarrow 0} P\left(|\epsilon X - a| < b\right) = 0$.

   (b) $\lim_{\epsilon \longrightarrow 0} \epsilon^2 \log P\left(|\epsilon X - a| < b\right) = -\dfrac{(a-b)^2}{2}$.

   (Note and hint. (b) improves (a) by giving a convergence rate. Try L'Hospital Rule for (b).)

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