**MA 519: Introduction to Probability Theory**
**August 2016, Qualifying Examination**

Your PUID: ____________________________

This examination consists of five questions, 20 points each, totaling 100 points. In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

1. 8 balls are randomly distributed into 4 boxes. Each ball is distributed independently from each other and uniformly among the 4 boxes.

   (a) What is the expected number of balls in a given box?

   (b) Let $N$ be the number of empty boxes. Compute $E[N]$ and $\mathrm{Var}(N)$.
   (Hint: use some appropriate indicator functions.)

   (c) What is the probability that there is at least one ball in every box.
   (Hint: use inclusion-exclusion principle.)

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2. Suppose that $0 \le X \le N$ are integer-valued random variables with the following joint distribution. For $\lambda > 0$ and $p \in (0,1)$ fixed parameters,

- the marginal distribution of $N$ is Poisson($\lambda$),
- the conditional distribution of $X$ given $N = n$ is Binomial($n$,$p$), i.e.

$$P(X = k \,|\, N = n) = \binom{n}{k} p^k (1-p)^{n-k}, \;\text{ for }\; k = 0, 1, 2, \ldots, n.$$

(a) Compute the marginal distribution of $X$, i.e., compute $P(X = k)$ for $k \ge 0$.

(b) Compute the conditional distribution of $N$ given $X = k$, i.e. compute $P(N = n \,|\, X = k)$.

(c) Compute the conditional mean $N$ given $X = k$, i.e. compute $E[N |\, X = k]$.
Hint: The conditional distribution calculated in part (b) is related to one of the standard probability distributions for which the mean and variance are well known.

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3. Prove the following identities.

(a) If $Z$ is a real-valued random variable with p.d.f. $f(z)$ and c.d.f. $F(t) = \int_{-\infty}^{t} f(z)\,dz$, then

$$E[Z] = \int_0^\infty (1 - F(t))\,dt - \int_{-\infty}^{0} F(t)\,dt$$

whenever both integrals on the right are finite.

(b) If $X$ is a non-negative, integer-valued random variable, then

$$E[X(X+1)] = \sum_{n=1}^{\infty} 2nP(X \geq n).$$

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4. Consider $X_\lambda$ to be a Poisson random variable with parameter $\lambda$.

   (a) Compute the moment generating function of $X_\lambda$, i.e., $M(s) = Ee^{sX_\lambda}$.

   (b) Show that $\dfrac{X_\lambda - \lambda}{\sqrt{\lambda}} \Longrightarrow_{\mathcal{D}} N(0,1)$ as $\lambda \longrightarrow +\infty$.
   Hint: recall the relationship between moment generating function and convergence in distribution.

   (c) Part (b) essentially says that for $n$ large, $\dfrac{X_n - n}{\sqrt{n}}$ approximately has the distribution of the standard normal random variable. Relate this to the statement of the classical Central Limit Theorem.

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5. (Estimation of the length of an interval.) Let $L > 0$ be some unknown but fixed length. Let $X_1, X_2, \ldots$ be a sequence of iid random variables uniformly distributed on $[0, L]$. The goal is to use the $X_i$'s to estimate $L$.

   (a) Let $A_n = 2\dfrac{X_1 + \cdots + X_n}{n}$. Show that $A_n$ is an unbiased estimator in the sense that $\mathrm{E}(A_n) = L$.

   (b) Let $B_n = \gamma_n \max\{X_1, X_2, \ldots, X_n\}$ where $\gamma_n$ is some number. Find the correct value of $\gamma_n$ such that $B_n$ is also an unbiased estimator, i.e. $\mathrm{E}(B_n) = L$.

   (Hint: find the distribution of $B_n$ first.)

   (c) Find $\mathrm{Var}(A_n)$ and $\mathrm{Var}(B_n)$.

   (d) Which estimator is "more superior"? Explain.

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