QUALIFYING EXAM COVER SHEET

August 2017 Qualifying Exams

Instructions: These exams will be "blind-graded" so under the student ID number please use your PUID

ID #: ____________________________
(10 digit PUID)

EXAM (circle one) **519** 523 530 544 553 554 562 571

For grader use:

Points _________ / Max Possible_________ Grade _________

MA 519 Qualifying Exam

Name:__________________________________

a) Legibly print your name above.

b) Do not open this test booklet until you are directed to do so.

c) You will have 120 min. to complete the exam. Budget your time wisely!

d) This test is closed book and closed notes. You may not use a calculator during this test.

e) Throughout the test, **show your work so that your reasoning is clear**. Otherwise no credit will be given.

f) If you need extra room, use the back of the pages. Just make sure I can follow your work.

1. Let $Z \sim \text{Exp}(\lambda)$.

a) Compute the distribution of $X = \lfloor Z \rfloor$, and compute $E[X]$.

b) Compute the p.d.f. of $Y = Z - \lfloor Z \rfloor$, and compute $E[Y]$.

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2. Suppose that $X \geq 1$ is a Geometric random variable with parameter $p \in (0, 1)$. That is,

$$P(X = k) = p(1-p)^{k-1}, \qquad \text{for } k \geq 1.$$

a) For any integer $m \geq 1$, compute the probability that X is a multiple of m.

b) Compute the probability that X is not divisible by any of the first three prime numbers (i.e., X is relatively prime to 2, 3 and 5).

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3. Compute $E[|X| + |Y|]$ in each of the following cases.

a) When (X, Y) are uniformly distributed on the unit circle $S^1 = \{(x, y) : x^2 + y^2 = 1\}$. *(Note: S^1 is NOT the unit disc)*

b) When X and Y are independent standard normal random variables.

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4. Recall that a permutation of n elements is a bijective function $\pi : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. The *descent* number of a permutation is

$$\mathrm{des}(\pi) = \#\{2 \le i \le n : \pi(i) < \pi(i-1)\}.$$

Example: if $n = 5$ *and* π *is the permutation given by*

n	1	2	3	4	5
$\pi(n)$	2	1	5	3	4

then $\mathrm{des}(\pi) = 2$.

Suppose that a permutation π of n elements is chosen uniformly at random from the $n!$ possible permutations, and $N = \mathrm{des}(\pi)$ is the descent number of π.

a) Compute $P(N > 0)$.

b) Compute $E[N]$.

c) Compute $\mathrm{Var}(N)$.

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5. Recall that if $X \sim \text{Poisson}(\lambda)$ for some $\lambda > 0$, then $P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$ for $k \geq 0$.

a) Prove that if $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$, and X and Y are independent, then $X + Y \sim \text{Poisson}(\theta)$ and compute the parameter θ in terms of λ and μ.

b) Use part a) and the law of large numbers to prove that

$$\lim_{n\to\infty} \sum_{k=0}^{\lfloor nt \rfloor} e^{-n}\frac{n^k}{k!} = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t > 1. \end{cases}$$

c) Prove that

$$\lim_{n\to\infty} \sum_{k=0}^{n} e^{-n}\frac{n^k}{k!} = \frac{1}{2}.$$

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