MA 519: Introduction to Probability Theory
Qualifying Examination – January 2017

Your PUID: Scores: (1) (2) (3) (4) (5) (T)

This examination consists of five questions, 20 points each, totaling 100 points. In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

1. Consider a disk of radius R: $B = \{(x, y) : x^2 + y^2 \le R^2\}$. A point $P = (X, Y)$ is chosen at random with uniform distribution in B. Let also D be the distance of P from the origin.
   (a) Find the marginal densities of X and Y.
   (b) Find the pdf of D.
   (c) Find $E(D)$, the expectation of D.

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2. (a) Let X and Y be two iid geometric random variables with parameter p. Find the conditional probability distribution of X given Y, i.e. find

$$P(X = i \Big| X + Y = j). \tag{1}$$

(b) Let S and T be two iid exponential random variables with parameter λ. Find the conditional probability density of S given $S + T$, i.e. find

$$p_{S|U}(s|u) \tag{2}$$

where $U = S + T$.

(c) Can you explain any relationship, if any, between your answers for (a) and (b)?

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3. An urn contains a large number of coins. Each coin gives a head with probability p. The value of p varies from coin to coin but is uniformly distributed in $[0, 1]$. Now a coin is selected at random. This *same* coin is used in the following question.

   (a) The coin is tossed once. What is the probability that the outcome is a head?

   (b) The coin is tossed twice. What is the probability that both outcomes are heads?

   (c) The coin is tossed n times. Let X be the number of heads obtained. Find the distribution of X, i.e. find $P(X = i)$.

   (d) The coin is kept being tossed until a head is obtained. Let N be the number of tossing needed. Find the distribution of N, i.e. find $P(N = n)$.

   (e) Find $E(N)$, the expectation of N.

   Note: The following integration identity might be useful: for any positive integers a, b,

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{(a-1)!(b-1)!}{(a+b-1)!}.$$

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4. A Gamma random variable with parameters α and λ has density given by:

$$p(x) = \frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}, \quad \text{for } x \geq 0.$$

The mean and variance of the above random variable are given by $\frac{\alpha}{\lambda}$ and $\frac{\alpha}{\lambda^2}$.

(a) Let W_α be a Gamma random variable with parameter α and λ. Explain why as $\alpha \longrightarrow \infty$ but with λ fixed, the random variable

$$\frac{W_\alpha - \frac{\alpha}{\lambda}}{\sqrt{\frac{\alpha}{\lambda^2}}}$$

converges in distribution to the standard normal distribution. You can assume that α only takes positive integer values.

For the remaining parts, let X_i's be a sequence of iid standard normal random variables.

(b) Find the pdf of X_1^2. How is it related to the Gamma distribution?

(c) Find the pdf of $X_1^2 + X_2^2 + \cdots X_n^2$.

(d) Give an approximation of $P(80 \leq X_1^2 + \cdots X_{100}^2 \leq 120)$. Express your answer in terms of the error function Φ.

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5. (a) Let X and Y be bi-variate normal random variables with joint probability density given as follows:

$$p(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\}$$

Find the conditional probability density of X and Y, i.e. find,

$$p_{X|Y}(x|y).$$

Relate your answer to some common distribution – be as quantitative as possible.

(Hint: use completing square.)

(b) Let $X \sim \mathcal{N}(\Theta, 1)$, i.e. normal random variable with mean Θ and variance 1. Now the actual value of Θ is not known but is distributed as $\mathcal{N}(0, 1)$, i.e. normal random variable with mean 0 and variance 1. The above information is expressed as:

$$p_{X|\Theta}(x|\theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\theta)^2}{2}\right), \text{ and } p_\Theta(\theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\theta^2}{2}\right).$$

An experiment is performed and an actual value X is obtained. Find the conditional probability distribution of Θ given $X = x$, i.e. find

$$p_{\Theta|X}(\theta|x).$$

Relate your answer to some common distribution – be as quantitative as possible.

(Hint: use Bayesian formula and completing square (again).)

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