MA 519: Introduction to Probability Theory
Qualifying Examination – August 2019

Your PUID: Scores: (1) (2) (3) (4) (5) (Total)

This examination consists of five questions, 20 points each, totaling 100 points. In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

Problem 1 The following is the axioms of probability.

(Axiom 1) $P(A) \geq 0$ for all event A.
(Axiom 2) $P(S) = 1$ for the sample space S.
(Axiom 3) (countable additivity) If $A_1, A_2, A_3...$ are mutually disjoint, i.e. $A_i \cap A_j = \phi$ for $i \neq j$, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

(a) (finite additivity) Using only the axioms, prove that if $A_1, A_2, ..., A_n$ are mutually disjoint, i.e. $A_i \cap A_j = \phi$ for $i \neq j$, then $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$.

(b) Using only the axioms, prove that if $A \subset B$, then $P(A) \leq P(B)$.

(c) Show that the countable additivity(Axiom 3) is equivalent to the following statement. We assume that the measure P is finitely additive.

$\lim_{n\to\infty} P(B_n) = 0$ for any sequence of nested classes $\{B_n\}$, i.e. $B_{n+1} \subset B_n$ and $\cap_{n=1}^{\infty} B_n = \phi$.

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Problem 2 Seven people take elevators on the first floor in a 10 story building. The elevator has an access to every floor. No one gets off the elevator on the first floor, since it does not make any sense. Also, no one stay in the elevator forever.

(a) What is the probability that there is only one floor where exactly two people get off?

(b) What is the probability that there is only one floor where exactly three people get off?

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Problem 3 Let $ABCD$ be a square with the area 1. Let $\alpha, \beta, \gamma, \delta$ be random points on $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ respectively. Let S be the area of the quadrangle $\alpha\beta\gamma\delta$. Find $E(S)$ and $Var(S)$.

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Problem 4 In Boilermaker dinning hall, the weight X of the sirloin steak follows exponential distribution with mean 12 oz. If the steak if less than 15 oz, it is already included in your meal plan, which means that you do not pay extra. If it is more than 15 oz, you need to pay extra $X - 15$ dollars. Let Y be the amount you need to pay more.

(a) Find the cdf of Y. In addition, plot the cdf.

(b) Find $E(Y)$.

(c) Find the probability that one pays extra dollars for the second time in 5th serving.

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Problem 5 X_is are i.i.d random variables from $Unif(0, \theta)$. Let $Y_n = \max\{X_1, X_2, ..., X_n\}$ and $Z_n = \min\{X_1, X_2, ..., X_n\}$.

(a) Find $E(Y_n)$.

(b) Let $W_n = n(\theta - Y_n)$. Find $F(w) = \lim_{n \to \infty} P(W_n \leq w)$.

(c) Find $P(X_{n+1} < Y_n)$.

(d) Find $P(X_{n+1} > Y_n | X_{n+1} > Z_n)$.