MA 519: Introduction to Probability Theory
Qualifying Examination – January 2019

Your PUID: Scores: (1) (2) (3) (4) (5) (Total)

This examination consists of five questions, 20 points each, totaling 100 points. In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

1. Suppose the number N of customers entering into a bank is given by a Poisson random variable with parameter λ. Each customer decides independently with probability p of going to teller number one and with probability $q = 1 - p$ of going to teller number two. Let X and Y be the number of customers going to teller number one and two, respectively.

   (a) Find the distributions of X and Y: $P(X = i)$ and $P(Y = j)$.

   (b) Find the joint distribution of X and Y: $P(X = i, Y = j)$.

   (c) Are X and Y independent?

   (d) Find the conditional distribution of X given N: $P(X = i|N = n)$.

   (e) Find the conditional distribution of N given X: $P(N = n|X = i)$.

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2. Let X and Y be two independent geometric random variables with parameter p.

   (a) Find the probability distribution function of $\min(X, Y)$: $P(\min(X, Y) = i)$.

   (b) Find the probability distribution function of $X + Y$; $P(X + Y = i)$.

   (c) Find the conditional distribition of X given $X + Y$: $P(X = i \Big| X + Y = j)$.

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3. Suppose at any time t the number of passengers arriving at a train station during the time interval $(0, t)$ is a Poisson random variable with parameter λt (i.e. λ can be interpreted as the "rate of arrival"). Suppose also that the arrival time of the train is uniformly distributed over $(0, T)$. Let the arrivals of the passengers and the train be independent of each other.

   Find the mean and variance of the number of passengers that can board the train.

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4. Let X be a standard normal random variable. Construct a random variable Y in the following way: toss a fair coin (which is independent of X) and set

$$Y = \begin{cases} X & \text{if the toss gives a head,} \\ -X & \text{if the toss gives a tail.} \end{cases}$$

In other words, Y is equally likely to equal to either X or $-X$.

(a) Find the pdf of Y and relate it to some common distribution.

(b) Find $E(XY)$, i.e the correlation between X and Y.

(c) Are X and Y independent?

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5. This question concerns finding the pdf of the quotient between random variables.

(a) Let the joint pdf of two continuous positive random variables X and Y be given by $p(x, y)$, for $x, y \geq 0$. Define $Z = \frac{Y}{X}$. Show that the pdf $p_Z(\cdot)$ of Z is given by

$$p_Z(z) = \int_0^\infty x p(x, zx)\, dx$$

(Hint: differentiate the cdf of Z.)

(b) Recall that Gamma distribution Gamma(α, λ) with parameter (α, λ) is given by

$$p_{\alpha,\lambda}(x) = \frac{1}{\Gamma(\alpha)} \lambda (\lambda x)^{\alpha-1} e^{-\lambda x} \quad \text{for } x > 0.$$

Let X and Y be two independent random variables distributed as Gamma(α, λ) and Gamma(β, μ). Find the pdf of $Z = \frac{Y}{X}$. *You do need to carry out all the integration to give a closed-form analytical expression.*

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