MA 519 Fall 2020
Qualifier

- You can use a calculator.
- This test is closed book and closed notes.
- You have 120 minutes.
- All problems have equal weight [10 points for each].
- Show your work.
- In order to get full credits, you need to give correct and simplified answers and explain in a comprehensible way how you arrive at them.
- Covid precaution: if you have questions about the exam, write them on a separate piece of paper and leave it on a dedicated table.
- **Good luck!**

Name:

Problem 1. A group of individuals containing b boys and g girls is lined up in random order; that is, each of the $(b+g)!$ permutations is assumed to be equally likely. What is the probability that the person in the ith position, namely $1, \ldots, i, \ldots, b+g$, is a girl? Be sure to define the sample space S corresponding to this experiment, as well as the probability $\mathbf{P}$ you are using on this sample space. Also include a mathematical description of the event you are considering as a subset of S.

Problem 2. Suppose we have 10 coins such that if the ith coin is flipped, heads will appear with probability $i/10$, for $i = 1, 2, \ldots, 10$. When one of the coins is randomly selected and flipped, it shows heads. What is the conditional probability that it was the fifth coin?

Problem 3. A restaurant can serve 75 meals for lunch. In practice, 20% of the reservations are canceled without further notice. **(a)** Model the situation, introducing a proper probability space and a family of random variables. **(b)** What is the maximal number of reservations which can be accepted in order to have a 90% chance to serve all incoming customers?

Problem 4. Let U and V be two independent real valued random variables, with respective densities f and g. **(a)** Show that the density of $Z = U + V$ is $f \star g$, where $f \star g(x) = \int_{\mathbb{R}} f(x-t)g(t)dt$. **(b)** Let $T_1, \dots, T_n$ be exponential random variables with parameter λ. Using question (a), compute the density of $S_n = T_1 + \cdots + T_n$.

Problem 5. (a) Let Y be a real valued continuous random variable. Prove that

$$\mathbf{E}[Y] = \int_0^\infty \mathbf{P}\left(Y > y\right)\, dy - \int_0^\infty \mathbf{P}\left(Y < -y\right)\, dy.$$

(b) For a non negative continuous random variable, variable prove that

$$\mathbf{E}\left[X^n\right] = n \int_0^\infty x^{n-1}\mathbf{P}\left(X > x\right)\, dx.$$

Problem 6. **(a)** Let Z be a standard normal variable. Show that $Y = Z^2$ follows Gamma$(\frac{1}{2}, \frac{1}{2})$.
(b) Consider X, Y respectively distributed as Gamma(α, λ) and Gamma(β, λ). Assume X and Y are independent. Show that $X + Y$ follows Gamma$(\alpha + \beta, \lambda)$.
(c) Let $\{X_i : i = 1, \ldots, n\}$ be i.i.d standard Normal random variables. Using the above questions (a) and (b), show that $Y = \sum_{i=1}^{n} X_i^2$ follows a χ^2 distribution with degree of freedom n.

Values of $\Phi(x)$ for some $x \geq 0$

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936