QUALIFYING EXAMINATION
JANUARY 1995
MATH 523

1. Consider the initial value problem

$$(1 - z^3)z_x + z_y = 0$$
$$z(x, 0) = f(x)$$

where $f \in C^1(\mathbb{R})$.

(a) Write down the equation that implicitly defines the solution z near the x-axis.

(b) ¿From your equation in (a), find formulas for z_x and z_y and verify that the PDE is satisfied.

(c) If $f(x) = -x^3$, decide whether or not shocks ever develop for $y \geq 0$, and justify your answer.

2. Let C be the parabola $y = x^2$ and consider the initial value problem

$$x^2 u_x - y^2 u_y + \cos(x - y)u = e^{xy}$$
$$u|_C = \varphi$$

where φ is a given continuous function defined on C. Prove that this problem has a unique solution in a neighborhood of the point $(1, 1)$.

3. (a) State carefully the maximum principle for harmonic functions.

(b) Let Ω be the upper-half of the unit ball in $\mathbb{R}^3$

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1, \quad z > 0\}$$

and let $u = 1/r$ where $r = (x^2 + y^2 + z^2)^{1/2}$. Show by direct computation that u is harmonic in Ω.

(c) Does u attain its maximum value in Ω? Does your answer contradict the maximum principle? Explain.

4. Let Ω be a domain in $\mathbb{R}^3$

(a) Define carefully the Green's function $G(\vec{r}', \vec{r})$ for the Dirichlet problem for Ω.

(b) Write down the formula for the solution of the Dirichlet problem

$$\begin{aligned} \triangle u = 0 \quad &\text{in} \quad \Omega \\ u = f \quad &\text{on} \quad \partial\Omega \end{aligned}$$

in terms of the Green's function.

(c) Prove that $G(\vec{r}', \vec{r}) \geq 0$ for all $\overline{r}\,', \vec{r} \in \Omega$, $\vec{r}\,' \neq \vec{r}$.

5. Consider the initial value problem for the wave equation in three space variables:

$$\begin{aligned} u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = 0; \quad &x \in \mathbb{R}^3, \ t > 0 \\ u(x,0) = \varphi(x); \quad &x \in \mathbb{R}^3 \\ u_t(x,0) = \psi(x); \quad &x \in \mathbb{R}^3 \end{aligned}$$

(a) Write down the formula for the solution $u(x,t)$ of the problem.

(b) If φ and ψ vanish outside a ball of radius 3 centered at the origin, find the set of points in $\mathbb{R}^3$ where you are sure that u vanishes when $t = 10$.

(c) If φ vanishes everywhere in $\mathbb{R}^3$, and

$$\psi(x) = \begin{cases} 0 & \text{for} \quad |x| < 1 \\ k & \text{for} \quad 1 \leq |x| \leq 2 \\ 0 & \text{for} \quad 2 < |x| \end{cases}$$

where k is a constant, find $u(0,t)$ for all $t \geq 0$. (Your answer should be explicit, no integrals).

6. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, and let $\vec{n}$ be the exterior unit normal on $\partial\Omega$

   (a) Suppose that $u(x_1, x_2, t)$ is of class C^2 in the closed half-cylinder

   $$(x_1, x_2) \in \overline{\Omega}, \quad t \geq 0,$$

   and that u satisfies the heat equation

   $$u_t - u_{x_1 x_1} - u_{x_2 x_2} = 0; \quad (x_1, x_2) \in \Omega, \quad t > 0$$

   and the boundary condition

   $$\frac{\partial u}{\partial n}(x_1, x_2, t) = -u(x_1, x_2, t); (x_1, x_2) \in \partial\Omega, \quad t \geq 0.$$

   Show that for any $T \geq 0$

   $$\int_\Omega u^2(x_1, x_2, T) dx_1 dx_2 \leq \int_\Omega u^2(x_1, x_2, 0) dx_1 dx_2$$

   (b) Consider the initial-boundary value problem

   $$\begin{aligned} u_t - u_{x_1 x_1} - u_{x_2 x_2} &= 0; \quad (x_1, x_2) \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial n}(x_1, x_2, t) &= -u(x_1, x_2, t); \quad (x_1, x_2) \in \partial\Omega, \quad t \geq 0 \\ u(x_1, x_2, 0) &= \varphi(x_1, x_2); \quad (x_1, x_2) \in \Omega. \end{aligned}$$

   Prove uniqueness of solution of this problem in the class of functions which are C^2 in the closed half-cylinder $(x_1, x_2) \in \overline{\Omega}, \quad t \geq 0$.

7. For each of the PDEs below give an explicit example of a solution which is in $C^2(\mathbb{R}^2)$ but not in $C^3(\mathbb{R}^2)$, if such a solution exists.

   (a) $u_{xx} + u_{yy} = 0$

   (b) $u_{xx} - u_{yy} = 0$

   (c) $u_{xx} - u_y = 0$.