QUALIFYING EXAMINATION

AUGUST 1996

MATH 523

1. Solve the initial value problem

$$z^2 z_x + x^3 y z_y = x^3 z$$
$$x = 2y,\ z = y^2,\ 0 < y < \infty$$

What problem occurs if the initial condition is given by

$$\sqrt{2}y = x^2, \quad \sqrt{2}z = x^2 \quad ?$$

2. Find the function $u(x,t)$ (in explicit form) which satisfies the equation

$$u_t + (u + u^2)u_x = 0, \quad 0 \le t, \quad -\infty < x < \infty$$

and initial condition

$$u(x,0) = \begin{cases} 0, & -\infty < x \le 0 \\ x, & 0 \le x < \infty \end{cases}$$

Do shocks appear? If so, where? If not what is the asymptotic behavior of the solution when $t \to \infty$.

3. Use Green's Second Identity to show that the only solution $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ to the bi-harmonic equation

$$\nabla^2(\nabla^2 \phi) = 0, \text{ in } \Omega$$

satisfying the boundary conditions

$$\varphi = 0, \quad \frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad \partial\Omega$$

is $\phi = 0$, where Ω is a bounded normal domain in $\mathbb{R}^3$.

4. Consider the Dirichlet problem

$$\nabla^2 u(r,\theta) = 0, \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$
$$u(1,\theta) = \frac{1}{1 - \epsilon \cos\theta}$$

where ϵ is a small parameter. Find an approximate solution $u_\epsilon(r,\theta)$ of the above problem which approximates the solution $u(r,\theta)$ in the disk to order ϵ^3, i.e.

$$\mathrm{Max}_{\substack{0\le r<1\\ 0\le\theta\le 2\pi}} |u(r,\theta)-u_\epsilon(r,\theta)| \le C\epsilon^3$$

where C is some constant.

5. Let Ω be the solid hemisphere $x_1^2+x_2^2+x_3^2 < a^2$, $x_3 > 0$ and $\delta\Omega$ its boundary. Find the Green's function $G(\mathbf{r}',\mathbf{r})$ for Laplace equations in Ω satisfying the Dirichlet boundary condition on $\delta\Omega$.
6. Let $\psi(s)$ be the function defined for $s \ge 0$ by

$$\psi(s) = \begin{cases} 1 & \text{for } 0 \le s \le 1 \\ 0 & \text{for } s > 1 \end{cases}.$$

Using Kirchoff's formula, derive equation

$$u(\mathbf{r}) = \frac{1-(t-r)^2}{4r}\psi(|t-r|), \quad r > 1$$

for the solution of the initial value problem,

$$\begin{gathered} \Delta u = u_{tt}, \quad \mathbf{r}\in\mathbb{R}^3, \quad t>0, \\ u(\mathbf{r},0) = 0, \quad \mathbf{r}\in\mathbb{R}^3, \\ u_t(\mathbf{r},0) = \psi(r), \quad \mathbf{r}\in\mathbb{R}^3. \end{gathered}$$

7. Using the solution

$$u(x,t) = \begin{cases} \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{t}}\exp\{-\frac{(x-\xi)^2}{4t}\}\varphi(\xi)d\xi, & t>0 \\ \varphi(x), & t>0 \end{cases}$$

to the heat conduction problem for an infinite rod, find the solution to the system

(i) $u_t - u_{xx} = 0, \quad 0 < x < \infty, \quad 0 < t$
(ii) $u(0,t) = u_0, \quad 0 < t$
(iii) $u(x,0) = 0, \quad 0 < x < \infty$

where u_0 is a constant.

8. Find the energy at time $t = 10$, for the system

$$\begin{gathered} u_{tt} - \nabla^2 u = 0 \quad x\in\Omega, \quad t>0 \\ u(x,0) = 1-(x_1^2+x_2^2+x_3^2), u_t(x,0) = 2, \quad x\in\Omega \\ u(x,t) = 0, \quad x\in\delta\Omega, \quad t>0 \end{gathered}$$

where Ω is the unit ball in $\mathbb{R}^3$.