QUALIFYING EXAMINATION
AUGUST 1997
MATH 523

1. Consider the initial value problem

$$zz_x + z_y = z$$
$$z(x, 0) = 3x$$

   (a) Use an existence and uniqueness theorem to show that the problem has a unique solution in a neighborhood of every point of the initial curve $y = 0$.
   (b) Solve the problem.

2. Let Ω be a domain in $\mathbb{R}^3$ and $\vec{V} = (P, Q, R)$ be a nonvanishing C^1 vector field in Ω. Suppose that $u \in C^1(\Omega)$, grad $u \neq \vec{0}$ in Ω, and that the level surfaces of u,

$$u(x, y, z) = c,$$

are the integral surfaces of $\vec{V}$ in Ω. Prove that if C is the integral curve of $\vec{V}$ passing through $(x_0, y_0, z_0) \in \Omega$, then C must lie on the integral surface of $\vec{V}$ passing through (x_0, y_0, z_0).

3. Prove uniqueness of solution of the initial-boundary value problem

$$u_{xx} - u_{tt} - au_t - bu = F(x, t); \quad 0 < x < L, \quad 0 \leq t$$
$$u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x); \quad 0 \leq x \leq L$$
$$u(0, t) = f(t), u_x(L, t) = g(t); \quad 0 \leq t$$

where a and b are nonnegative constants, and F, φ, ψ, f, and g are sufficiently smooth functions. Assume that $u(x, t)$ is C^2 for $0 \leq x \leq L$ and $0 \leq t$.

4. Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and let $\vec{n}$ be the exterior unit normal vector on $\partial\Omega$.
   (a) Define carefully the Green's function $G(\vec{r}\,', \vec{r})$ for the Dirichlet problem for Ω.
   (b) Write down the formula for the solution of the Dirichlet problem

$$\nabla^2 u = 0 \quad \text{in} \quad \Omega$$
$$u = f \quad \text{on} \quad \partial\Omega$$

   in terms of the Green's function.
   (c) Show that for each fixed $\vec{r}$ in Ω, $\frac{\partial}{\partial n} G(\vec{r}\,', \vec{r}) \leq 0$, for $\vec{r}\,' \in \partial\Omega$.
   (d) Show that for each $\vec{r} \in \Omega$,

$$-\int_{\partial\Omega} \frac{\partial}{\partial n} G(\vec{r}\,', \vec{r}) d\sigma = 1.$$

5. Consider the initial-boundary value problem for the heat equation,

$$
\begin{aligned}
u_t - u_{xx} &= 0; \quad 0 < x < L, \quad 0 < t \\
u_x(0,t) = u_x(L,t) &= 0; \quad 0 \le t \\
u(x,0) &= \begin{cases} 0 \text{ for } 0 \le x < \frac{L}{2} \\ 100 \text{ for } \frac{L}{2} \le x \le L \end{cases}.
\end{aligned}
$$

(a) Find the series solution of the problem.
(b) Does the series solution converge uniformly when $t = 0$? Explain.
(c) Prove that the solution is C^∞ when $t > 0$.

6. For each of the PDEs below, construct a solution which is in $C^2(R^3)$ but not in $C^3(\mathbb{R}^3)$. If this is not possible, explain why.

(a) $u_{xx} + u_{yy} - u_{zz} = 0, \quad (x, y, z) \in \mathbb{R}^3$
(b) $u_{xx} + u_{yy} + u_{zz} = 0, \quad (x, y, z) \in \mathbb{R}^3$
(c) $u_{xx} + u_{yy} - u_z = 0, \quad (x, y, z) \in \mathbb{R}^3$

7. Consider the linear first order PDE in two variables,

$$a(x, y)u_x + b(x, y)u_y = 0$$

where a and b are C^1 and do not vanish simultaneously. Prove that if C is a characteristic curve of the PDE, then a solution $u(x, y)$ of the PDE must be constant on C.

8. State carefully the theorem on the domain of dependence inequality for the wave equation in two space variables,

$$u_{xx} + u_{yy} - u_{tt} = 0.$$