QUALIFYING EXAMINATION

JANUARY 1997

MATH 523

(17 pts) 1. (a) Determine the regions where the equation

$$y^2 u_{xx} + \frac{3}{2} xy u_{xy} - x^2 u_{yy} + x u_x + y u_y = 0$$

is hyperbolic, parabolic or elliptic.

(b) Find the equations of the characteristic curves.

(17 pts) 2. Find the solution to the initial-value boundary-value problem

$$\begin{aligned} &v_t = v_{xx} + f(x,t), && 0 < x < \infty, && t > 0 \\ &v(x,0) = 0, && 0 < x < \infty, && \\ &v(0,t) = 0, && && t > 0 \end{aligned}$$

where $f(x,t)$ is a bounded continuous function that vanishes at $x = 0$.

(17 pts) 3. (a) If Ω is a bounded normal domain in $\mathbb{R}^3$, with boundary $\partial\Omega$, and $q(x) \in C(\Omega)$, $q(x) \geq 0$, derive an energy integral for the system

$$\begin{aligned} &u_{tt} - \nabla^2 u + q(x) u = 0, && x \in \Omega, \quad t > 0 \\ &u(x,t) = 0, && x \in \partial\Omega \end{aligned}$$

and show that the resulting energy integral is constant in time.

(b) Use the results of part (a) to show that small errors in the data $g(x)$, $h(x)$ of the system

$$\begin{aligned} &u_{tt} - \nabla^2 u + q(x) u = 0, && x \in \Omega, \quad t > 0 \\ &u(x,t) = 0, && x \in \partial\Omega, \quad t > 0 \\ &u(x,0) = g(x), \quad u_t(x,0) = h(x), && x \in \Omega \end{aligned}$$

produce small changes in the solution. Assume that $g(x) \in C^1(\Omega)$, $h(x) \in C(\Omega)$.

(18 pts) 4. Find the explicit form of the solution $u(x,t)$ for $t > 0$ for the system

$$u_t + u^2 u_x = 0, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = \begin{cases} x, & 0 \le x \\ 0, & x < 0. \end{cases}$$

Do shocks develop for $t > 0$?

(17 pts) 5. Let Ω be the hemisphere, $\Omega = \{x \in \mathbb{R}^3 \mid |x| < a, x_3 > 0\}$ and let $u(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of the Dirichlet problem

$$\nabla^2 u = 0, \quad x \in \Omega$$
$$u(x) = \begin{cases} g(x), & |x| = a, \quad x_3 > 0 \\ 0, & |x| < a, \quad x_3 = 0 \end{cases}$$

where $g(x)$ is a continuous function on the surface $|x| = a$, $x_3 > 0$, with $g(x) = 0$ at $x_3 = 0$. Find the explicit form of the solution $u(x)$, expressed in terms of an integral over the surface $\partial\Omega$.

(14 pts) 6. (a) Show, using the appropriate Green's identity that the Robin problem

$$\triangle u = 0, \qquad x \in \Omega$$
$$\frac{\partial u}{\partial n} + \alpha u = h(x), \qquad x \in \partial\Omega$$

has a unique $C^2(\Omega) \cap C^1(\overline{\Omega})$ solution when $\alpha > 0$. Here Ω is a bounded normal domain in $\mathbb{R}^3$.

(b) Find the solution to the problem in part (a) when Ω is the disk $x^2 + y^2 < 1$, and the boundary data $h(x)$ expressed in polar coordinates (r, θ) is given by $h = \cos^2\theta$.