MA-523 Qualifying Exam, August 10, 2001

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Name:

1. (20 pts)

(a) Prove that the function

$$K(x,\xi) = \frac{1}{2}|x-\xi|, \quad x \in \mathbf{R},\ \xi \in \mathbf{R},$$

is a fundamental solution for the operator $P = \dfrac{d^2}{dx^2}$ with pole ξ.

(b) Find the Green's function $G(x,\xi)$ of the operator P above in the interval $[0,1]$, i.e., find $G(x,\xi)$, for $x \in [0,1]$, $\xi \in (0,1)$, such that

$$\frac{d^2}{dx^2}G(x,\xi) = \delta_\xi(x), \quad G(0,\xi) = G(1,\xi) = 0, \quad \forall \xi \in (0,1).$$

Plot $G(x,\xi)$ as a function of x for a fixed $\xi \in (0,1)$, for example $\xi = 1/3$.

(c) Using the result in (b) express the solution of the problem

$$u''(x) = f(x), \quad u(0) = u(1) = 0,$$

where f is integrable over $[0,1]$, as certain integral(s) of f. Use this to solve

$$u'' = e^x \quad \text{in } (0,1), \quad u(0) = u(1) = 0 \tag{1}$$

and then compare your answer with the solution of (1) that can be found directly by integration.

2. (15 pts) Solve the following initial value problem for the transport equation

$$\begin{cases} \partial_t u(t,x) &= -v\cdot\nabla_x u(t,x) - a(x)u(t,x), & t \in \mathbf{R},\ x \in \mathbf{R}^n, \\ u(0,x) &= f(x), & x \in \mathbf{R}^n. \end{cases} \tag{2}$$

Here $0 \neq v \in \mathbf{R}^n$ is a fixed vector, $a(x)$ and $f(x)$ are given continuous functions of $x \in \mathbf{R}^n$, and $\nabla_x = (\partial_{x_1}, \dots, \partial_{x_n})$.

3. (17 pts)

(a) Find the type of the equation

$$yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0, \quad x \neq y \tag{3}$$

and make a change of variables reducing (3) into its normal form.

(b) Find the general solution of (3). Write down the solution in the original (x,y) variables.

4. (16 pts) Let $K \in \mathbf{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary, let $D = \mathbf{R}^n \setminus K$ be its exterior, and consider the "exterior" Dirichlet problem

$$\Delta u = 0 \quad \text{in } D, \qquad u|_{\partial D} = f, \tag{4}$$

where $f \in C(\partial D)$.

(a) Prove that the solution of (4) is unique in the class of functions $u \in C(\bar{D}) \cap C^2(D)$ satisfying the limit $\lim_{|x|\to\infty} u(x) = 0$.

(b) Show that without the limit condition, the uniqueness may fail. To this end, assume that $D = \{x;\ |x| > 1\}$ is the complement of the unit ball in $\mathbf{R}^3$ and $f(x) = 1$, and find explicitly two different solutions of (4) in $C(\bar{D}) \cap C^2(D)$ such that at least one of them does not tend to 0, as $|x| \to \infty$.

5. (15 pts) Let $D \subset \mathbf{R}^n$, $n \geq 2$ be an open set and let $u \in C^2(D)$. Prove that if for any sphere S belonging to D together with its interior, we have

$$\int_S \frac{\partial u}{\partial \nu} dS_x = 0,$$

then u is harmonic in D. Here ν is the outer normal to S as usual.

6. (17 pts)

(a) Let Q be the square $Q = \{(x,y) \in \mathbf{R}^2;\ 0 < x < \pi, 0 < y < \pi\}$. Solve the following heat transfer problem

$$\begin{cases} u_t - \Delta u &= 0, & \text{for } t > 0,\ (x,y) \in Q, \\ u|_{\partial Q} &= 0, & \text{for } t > 0, \\ u|_{t=0} &= xy(\pi - y), & \text{for } (x,y) \in Q, \end{cases}$$

where $\Delta = \partial_x^2 + \partial_y^2$.

(b) Prove that $0 < u(t,x,y) < \pi^3/4$ for all $t > 0$ and $(x,y) \in Q$.

(c) Prove that there exists a constant $C > 0$ such that $0 < u(t,x,y) \leq Ce^{-2t}$ for $t \geq 0$ and $(x,y) \in Q$.