QUALIFYING EXAMINATION
JANUARY 2004
MATH 523 - Prof. Phillips

1. Consider the Cauchy problem

$$u_x - 2xu_y = u \text{ for } (x, y) \in \mathbb{R}^2$$
$$u(x, x^2) = 2x \text{ for } x \in \mathbb{R}$$

a) Quote a theorem to show that there is a unique solution in a neighborhood of every point on $y = x^2$ except $(0, 0)$. Indicate why the theorem does not apply to $(0, 0)$.

b) Find the solution in a neighborhood of $(1, 1)$.

2. Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary. Let $u \in C^2(\overline{\Omega})$ be such that $u \not\equiv 0$ and solves

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

for some constant λ. Use Green's First Identity to establish the following.

a) Prove that $\lambda > 0$.

Let $v \in C^2(\overline{\Omega}), v \not\equiv 0$, and solves

$$\begin{aligned} -\Delta v &= \mu v && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $\mu \neq \lambda$.

b) Show that $\displaystyle\int_{\Omega} uv dx = 0$.

3. Consider the Cauchy problem

$$au_{xx} + bu_{xy} + cu_{yy} + d = 0 \text{ on } \Omega \subset \mathbb{R}^2,$$

$$\begin{aligned} u(0,y) &= f(y), \text{ on } \{0\} \times \mathbb{R} \cap \Omega, \\ u_x(0,y) &= g(y), \end{aligned}$$

where a, b, c, d are functions of x, y, u, u_x, u_y and Ω is an open set. Carefully state the conditions on a, b, c, d, f, and g so that the Cauchy-Kovelevsky Theorem can be applied in a neighborhood of $(0, 0)$. State the result as well.

4. Let $\Omega \subset \mathbb{R}^2$ be an open connected bounded set with a smooth boundary. Suppose Ω has the symmetry:

$$(x, y) \in \Omega \qquad \text{if and only if} \qquad (-x, y) \in \Omega.$$

Let $u(x, y) \in C^2$ be harmonic in Ω. Set $v(x, y) = u(-x, y)$ for $(x, y) \in \Omega$.

a) Show that v is harmonic in Ω.

Let $u \in C^2(\overline{\Omega})$ satisfying

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

where $g(x, y) = g(-x, y)$ for $(x, y) \in \partial\Omega$.

b) Prove that $u(x, y) = u(-x, y)$ for $(x, y) \in \Omega$ and show that

$$u_x(0, y) = 0 \qquad \text{if } (0, y) \in \Omega.$$

5. Let $h(x,t)$ be a bounded C^1 function.

a) For each τ fixed give the formula for $v(x,t;\tau)$ solving

$$\begin{aligned} v_{tt} - v_{xx} &= 0 && \text{for } x \in \mathbb{R}, \quad t > \tau, \\ v(x,\tau;\tau) &= 0, && \text{for } x \in \mathbb{R}, \\ v_t(x,\tau;\tau) &= h(x,\tau) && \text{for } x \in \mathbb{R}. \end{aligned}$$

b) Use Duhamel's principle to solve

$$\begin{aligned} u_{tt} - u_{xx} &= h(x,t) && \text{for } x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= 0 && \text{for } x \in \mathbb{R}, \\ u_t(x,0) &= 0 && \text{for } x \in \mathbb{R} \end{aligned}$$

in terms of v. Verify, by direct substitution, that the formula solves the Cauchy problem.

c) For given (x_0, t_0) what is the domain of dependence of u on h?

6. Let g be a continuous function with compact support in $\mathbb{R}^n$. Write a formula for the bounded solution to

$$\begin{aligned} u_t - \Delta u = 0 \qquad & \text{for } x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x) \qquad & \text{for } x \in \mathbb{R}^n. \end{aligned}$$

Prove that

$$\lim_{t \to \infty} u(x,t) = 0 \quad \text{where the convergence is uniform in } x.$$