QUALIFYING EXAMINATION

AUGUST 2005 MATH 523 – A. PETROSYAN AND P. STEFANOV

Each problem is worth 20 points.

1. (a) Find *a solution* of the Cauchy problem

$$yu_x + xu_y = xy, \qquad u = 1 \quad \text{on} \quad S := \{x^2 + y^2 = 1\}.$$

(b) Is the solution unique in a neighborhood of the point $(1, 0)$? Justify your answer.

2. Consider the second order PDE in $\{x > 0, y > 0\} \subset \mathbb{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

(a) Classify the equation and reduce it to the canonical form.

(b) Show that the general solution of the equation is given by the formula

$$u(x, y) = F(xy) + \sqrt{xy}\, G(x/y).$$

3. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy$$

is a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if f is radial (i.e. $f(y) = f(|y|)$) and supported in $B_R := \{|x| < R\}$, then

$$u(x) = c\,\Phi(x), \quad \text{for any} \quad x \in \mathbb{R}^n \setminus B_R,$$

where $c = \int_{\mathbb{R}^n} f(y) dy$.

Hint: Use spherical (polar) coordinates and the mean value property.

4. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha\, u_t = 0 & \text{in} \quad \mathbb{R}^2 \times (0,\infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}^2, \end{cases}$$

where $g, h \in C_0^\infty(\mathbb{R}^2)$ and $\alpha \geq 0$ is a constant.

(a) Show that for an appropriate choice of constants λ and μ the function

$$v(x_1, x_2, x_3, t) := \exp(\lambda\, t + \mu\, x_3)\, u(x_1, x_2, t)$$

solves the 3-dimensional wave equation $v_{tt} - \Delta v = 0$.

(b) Use (a) to prove the following domain of dependence result: for any point $(x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$ the value $u(x_0, t_0)$ is uniquely determined by the values of g and h in $\overline{B_{t_0}(x_0)} := \{x \in \mathbb{R}^2 : |x - x_0| \leq t_0\}$. (You may use the corresponding result for the wave equation without proof.)

5. Let $u(x,t)$ be a bounded solution of the heat equation $u_t = u_{xx}$ in $\mathbb{R} \times (0,\infty)$ with the initial condition

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R},$$

where $u_0 \in C^\infty(\mathbb{R})$ is 2π-periodic, i.e. $u_0(x+2\pi) = u_0(x)$. Show that

$$\lim_{t\to\infty} u(x,t) = a_0,$$

uniformly in $x \in \mathbb{R}$, where

$$a_0 := \frac{1}{2\pi}\int_0^{2\pi} u_0(x)dx.$$