August 2007

MATH 523

QUALIFYING EXAM

SHOW ALL YOUR WORK. EACH PROBLEM IS WORTH 20 POINTS.

1) (a) Show that inversion with respect to $S(0,1)$, i.e. the mapping $\xi \longrightarrow \xi^* = \frac{\xi}{|\xi|^2}$, maps the quarter plane $\Omega = \{(x,y) \in \mathbb{R}^2, x > 1, y > 1\}$ onto the lens shaped domain Ω^* bounded by the two circles

$$\left(x - \tfrac{1}{2}\right)^2 + y^2 = \tfrac{1}{4}, \qquad x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}.$$

(b) Solve the Dirichlet boundary value problem for Laplace's equation in Ω^*, with $u^* = 1$ on the top boundary of the lens, and $u^* = 0$ on the bottom boundary of the lens.

[Hint: The solution corresponding to the Dirichlet problem in the quarter space $x > 1$, $y > 1$ is easily found to be

$$u(x,y) = \tfrac{2}{\pi}\,\theta = \tfrac{2}{\pi}\,\arctan\,\tfrac{y-1}{x-1},$$

where θ is the polar angle with (1,1) as the center of the polar coordinate system.]

2) Consider the initial value problem for the conservation law $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$ for $x \in \mathbb{R}$, $t > 0$, with initial condition $u(x,0) = h(x)$ for $x \in \mathbb{R}$, where f is strictly convex (i.e. $f''(u) > 0$ for all u). Show that, if there is a point $\bar{x}$ at which h is decreasing, then the initial value problem does not admit a classical solution, i.e. one that is continuously differentiable for $x \in \mathbb{R}$ and $t > 0$.

[Hint: Use the implicit function theorem to compute the partial derivatives u_t and u_x.]

3) Let $L > 0$ and consider the following initial-boundary value problem:

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < L, 0 < t, \\ u(x,0) = \varphi(x), & 0 \le x \le L, \\ u(0,t) = 0, u(L,t) = 1; & 0 \le t, \end{cases}$$

where φ is continuous with sectionally continuous derivative on $[0,L]$, $\varphi(0) = 0$, and $\varphi(L) = 1$.

(a) Find the solution $u(x,t)$ of the problem.

(b) Is the solution C^∞ for $t > 0$? Justify your answer.

(c) Find $\lim_{t\to\infty} u(x,t)$.

4) Find the solution of the initial value problem

$$\begin{cases} u_{tt} - u_{xx} + u = 0, & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = f(x), & \text{in } \mathbb{R}, \\ u_t(x,0) = g(x), & \text{in } \mathbb{R}, \end{cases}$$

where $f, g \in C_0^\infty(\mathbb{R})$.
[Hint: Let $v(x,y,t) = h(y)u(x,t)$, and find h so that v solves the two-dimensional wave equation.]

5) State the Cauchy-Kovalevskaya Theorem. Under what assumptions on f (if any) does this theorem guarantee the existence of a unique real analytic solution to the Cauchy problem

$$\begin{cases} (\sin u)u_x - u_y + uu_z = \tan u, \\ u(x,y,x^2) = f(x), \end{cases}$$

near the point $(2,1,4)$? Justify your answer.