Math 523
Qualifying Examination
January 3, 2007

Name..

I. D. no. ....

Problem	Score	Max. pts.
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1. 1) Let $\mathbb{S}^{n-1} = \{\omega \in \mathbb{R}^n \mid |\omega| = 1\}$ be the unit sphere centered at the origin. Prove that the function $u(x,t) = e^{i\sqrt{\lambda}t}\phi(x)$, where $\phi \in C^\infty(\mathbb{R}^n)$ is defined by

$$\phi(x) \; = \; \int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}\, <x,\omega>} \, d\sigma(\omega) \; , \qquad \lambda > 0 \; , \quad x \in \mathbb{R}^n \; ,$$

solves the wave equation $\Box u = \Delta u - u_{tt} = 0$ in $\mathbb{R}^{n+1}$. Here, $d\sigma$ denotes the $(n-1)$-dimensional surface measure on $\mathbb{S}^{n-1}$, and $i^2 = -1$.

2) Find an explicit formula for $u(x,t)$ when $n = 3$.

Problem 2. Solve the Cauchy problem for the nonlinear equation

$$\begin{cases} u_t + u^2 u_x &= 0 \ , \\ u(x,0) &= x \ . \end{cases}$$

Find the region in the (x,t)-plane where the solution develops shocks (i.e., discontinuities).

Problem 3. Use Fourier transform to solve the Cauchy problem

$$\begin{cases} \Delta u - u - u_t \;=\; 0\,, \quad \text{in } \mathbb{R}^n \times (0,\infty)\,, \\ u(x,0) \;=\; 1 \text{ if } |x| \leq \; 1\,,\; u(x,0) \;=\; 0\,,\; \text{if } |x| > 1\,. \end{cases}$$

Problem 4. Let

$$\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{36} + \frac{y^2}{9} + \frac{z^2}{25} < 1 \right\} \ ,$$

and assume that $u \in C^2(\overline{\Omega})$ solves the problem

$$\begin{cases} \Delta u \ = \ -1 \ , & \text{in} \ \ \Omega \ , \\ u \ = \ 0 \ , & \text{on} \ \ \partial\Omega \ . \end{cases} \tag{1}$$

Prove that

$$\frac{3}{2} \ \leq \ u(0,0,0) \ \leq \ 6 \ .$$

Problem 5. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1, and $b \in C^1(\mathbb{R}^{n+1};\mathbb{R}^n)$. Prove that there exists a constant $C = C(\Omega, n) > 0$ sufficiently small such that, if $||(div\ b)^-||_{L^\infty(\mathbb{R}^{n+1})} \leq C$, then there exists at most one solution $u \in C^{2,1}(\overline{\Omega} \times [0,\infty))$ to the problem

$$\begin{cases} \Delta u + \ < b, Du > \ - \ u_t \ = \ 0 \qquad\qquad \text{in} \quad \Omega \times [0,\infty) \ , \\ u(x,0) \ = \ \phi(x) \ , \quad x \in \Omega \ , \quad u(x,t) \ = \ \psi(x,t) \ , \ (x,t) \in \partial\Omega \times (0,\infty) \ . \end{cases}$$

Here, for a given $a \in \mathbb{R}$, a^+, a^- respectively denote its positive and negative part, so that $a = a^+ - a^-$.

Note: You can assume the validity of the Poincaré inequality

$$\int_\Omega f(x)^2 dx \ \leq \ C^*(n)(diam\ \Omega)^2 \int_\Omega |Df|^2 dx \ ,$$

where $C^*(n) > 0$ depends only on the dimension n, and f is an arbitrary function in $C^1(\overline{\Omega})$ such that $f = 0$ on $\partial\Omega$.