Math 523
Qualifying Examination
August , 2008
Prof. N. Garofalo

Name..

I. D. no. ....

Problem	Score	Max. pts.
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1. 1) Let $\Omega \subset \mathbb{R}^n$ be an open set and consider a sequence $\{f_k\}_{k\in\mathbb{N}}$, $f_k \in C^2(\Omega)$, of harmonic functions in Ω such that $0 \le f_k \le f_{k+1}$ and for which

$$f(x) \overset{def}{=} \underset{k\in\mathbb{N}}{sup}\ f_k(x) < \infty, \quad \text{for every } x \in \Omega.$$

Prove that f is harmonic in Ω.

Problem 2. Let u be a solution of the initial value problem for the nonlinear equation

$$\begin{cases} u_t + uu_x &= 0 \ , \\ u(x,0) &= x \ . \end{cases}$$

Find the region in the (x,t)-plane where the solution develops shocks (i.e., discontinuities).

Problem 3. Use Fourier transform to solve the Cauchy problem

$$\begin{cases} u_{xx} - u_{tt} \;=\; 0\;, \quad \text{in } \mathbb{R}\times(0,\infty)\;, \\ u_t(x,0) \;=\; 1 \text{ if } |x| \le \;1\;,\; u_t(x,0) \;=\; 0 \;\text{ if } |x| > 1\;, \quad u(x,0) = 0. \end{cases}$$

Problem 4. **(i)** Let $\mathbb{S}^{n-1} = \{\omega \in \mathbb{R}^n \mid |\omega| = 1\}$ be the unit sphere centered at the origin, and define

$$\phi(x) \;=\; \int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}\,<x,\omega>}\, d\sigma(\omega)\ , \qquad \lambda > 0\ , \quad x \in \mathbb{R}^n\ ,$$

where, $d\sigma$ denotes the $(n-1)$-dimensional surface measure on $\mathbb{S}^{n-1}$, and $i^2 = -1$. Prove that the function $u(x,t) = e^{-\lambda t}\phi(x)$, solves the heat equation $Hu = \Delta u - u_t = 0$ in $\mathbb{R}^{n+1}$.

(ii) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and suppose that $u \in C^3(\Omega \times (0,\infty))$ be a solution to $Hu = \Delta u - u_t = 0$ in $\Omega \times (0,\infty)$. Prove that the function $f = |Du|^2 + u_t^2$ cannot attain a maximum at a point (x_0, t_0), with $x_0 \in \Omega$ and $t_0 > 0$, unless $u \equiv \textit{constant}$ in $\Omega \times (0, t_0)$.

Problem 5. For $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and a given function ϕ, indicate $\phi_{x_i} = \frac{\partial \phi}{\partial x_i}$, $i = 1, ..., n$. Solve the non-homogeneous initial value problem

$$\begin{cases} \phi_t + \phi_{x_1} - \phi_{x_n} = 2t + |x|^2, & \text{in } \mathbb{R}^n \times (0, \infty), \\ \phi(x, 0) = x_1^2 - x_n^2, & x \in \mathbb{R}^n. \end{cases}$$