Math 523
Qualifying Examination
August 18, 2009
Prof. N. Garofalo

Name..

I. D. no. ....

Problem	Score	Max. pts.
1		20
2		20
3		20
4		30
5		40
Total		130

Problem 1. Let ϕ be a continuous function on $\mathbb{R}^n$ with compact support. Prove that if ϕ is spherically symmetric, i.e. there exists $\phi^* : [0,\infty) \to \mathbb{R}$ such that $\phi(x) = \phi^*(|x|)$, $x \in \mathbb{R}^n$, then the solution of the homogeneous Cauchy problem

$$\begin{cases} \Delta f - f_t = 0, & \text{in } \mathbb{R}^n \times (0,\infty), \\ f(x,0) = \phi(x), & x \in \mathbb{R}^n, \end{cases}$$

is a function $f(x,t)$ having spherical symmetry in the variable $x \in \mathbb{R}^n$ as well.

Problem 2. Consider the function

$$f(x,t) = \int_0^1 \frac{t}{t^2 + (x-y)^2} dy \ , \qquad (x,t) \in \mathbb{R}^2_+ = \mathbb{R} \times (0, \infty) \ .$$

Motivating your answer decide which of the following statements is true:

A. $f \in C^1(\mathbb{R}^2_+)$, but $f \notin C^2(\mathbb{R}^2_+)$

B. $f \in C^2(\mathbb{R}^2_+)$ and $f_{xx} + f_{tt} = -2f$ in $\mathbb{R}^2_+$

C. $f \in C^2(\mathbb{R}^2_+)$ and $f_{xx} + f_{tt} = -f$ in $\mathbb{R}^2_+$

D. $f \in C^2(\mathbb{R}^2_+)$ and $f_{xx} + f_{tt} = 0$ in $\mathbb{R}^2_+$

E. $f \in C^2(\mathbb{R}^2_+)$ and $f_{xx} + f_{tt} = 2$ in $\mathbb{R}^2_+$

Problem 3. Let $\phi \in C(\mathbb{R}^3)$ be a function with compact support. Solve the Cauchy problem

$$\begin{cases} \Delta f - \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_3} - \frac{\partial f}{\partial t} = 0, & \text{in } \mathbb{R}^3 \times (0, \infty), \\ f(x, 0) = \phi(x), & x \in \mathbb{R}^3. \end{cases}$$

Problem 4. Let $f \in C^2(\mathbb{R}^n)$ be a solution in $\mathbb{R}^n$ of the equation $\Delta f = |x|^3$. Prove that an estimate such as

$$|f(x)| \ \le \ C \ (1 + |x|^{4+\epsilon}) \ , \qquad\qquad x \in \mathbb{R}^n \ ,$$

is impossible for $C \ge 0$ and $0 \le \epsilon < 1$.

Problem 5. Solve the Cauchy problems:

$$\begin{cases} \Delta f - f_{tt} \;=\; |x|^2 & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ f(x,0) \;=\; 0, \quad f_t(x,0) \;=\; 0 & x \in \mathbb{R}^3. \end{cases}$$

$$\begin{cases} \Delta f - f_{tt} \;=\; 0 & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ f(x,0) \;=\; 0, \quad f_t(x,0) \;=\; \mathbf{1}_{B(0,R)}, & x \in \mathbb{R}^3 \end{cases}$$

where $\mathbf{1}_{B(0,R)}$ denotes the indicator function of the ball in $\mathbb{R}^3$ centered at the origin with radius $R > 0$.