Math 523
Qualifying Examination
August 9, 2010
Prof. N. Garofalo

Name...

I. D. no. ....

Problem	Score	Max. pts.
1		20
2		20
3		30
4		30
5		30
Total		130

Problem 1. Let E be a regular hexagon centered at the origin of the plane $\mathbb{R}^2$. Let f be the harmonic function in E with boundary value 1 on one of the sides of E and 0 on each of the remaining five sides. What is the value of f at the origin? Explain your reasoning.

Problem 2. Consider the Cauchy problem for the wave equation

$$\begin{cases} f_{xx} - f_{tt} = 0, & \text{in } \mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}_+, \\ f(x,0) = \phi(x), \quad f_t(x,0) = \psi(x), & x \in \mathbb{R}. \end{cases}$$

Suppose that ϕ, ψ vanish outside of the interval $[-1, 1]$. In which region of the upper half-plane $\mathbb{R}^2_+$ one is guaranteed that the solution f vanishes identically?

Problem 3. Let $f \in C^1(\mathbb{R}^2)$ be a solution of the first-order equation

$$f_t + f f_x = 0, \qquad (x,t) \in \mathbb{R}^2.$$

Prove that $f \equiv$ constant.

Hint: By analyzing the characteristic lines of f starting at $(x,0)$ prove that the function $x \to f(x,0)$ must be constant on $\mathbb{R}$

Problem 4. For a given $\alpha > 0$ let $f \in C^2(\mathbb{R}^n)$ be a solution in $\mathbb{R}^n$ of the equation $\Delta f = |x|^\alpha$. Prove that for any $\beta < \alpha + 2$ one must have

$$\limsup_{|x|\to\infty} \frac{|f(x)|}{|x|^\beta} = +\infty.$$

Problem 5. Given $x_0 \in \mathbb{R}^n$ and $R > 0$ consider the homogeneous Cauchy problem

$$\begin{cases} \Delta f - f_t = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ f(x, 0) = \mathbf{1}_{B(x_0,R)}(x), & x \in \mathbb{R}^n, \end{cases}$$

where $\mathbf{1}_{B(x_0,R)}$ is the indicator function of the ball $B(x_0, R) = \{x \in \mathbb{R}^n \mid |x - x_0| < R\}$. Prove that there exists $0 < A < 1$, depending only on $n \in \mathbb{N}$, but not on x_0 or R, such that

$$f(x_0, AR^2) \geq \frac{1}{2}.$$