Qualifying Exam - Math 523 - January 2010 - P. Bauman

Instructions: Show your reasoning in all problems.

1. Consider the following Cauchy problem:

$xu_x - yu_y = u - 1,$
$u(x, x) = 1 + x^3.$

a) (8 pts.) At what values x_0 is there a unique C^1 solution in a neighborhood of (x_0, x_0)? Cite a theorem to support your answer.

b) (12 pts.) Find the solution near the points (x_0, x_0) found in 1.a).

2.) (20 pts.) Let D be a bounded smooth domain in $\mathbf{R^n}$. Assume that u is a given function in $C^3(D) \bigcap C^2(\overline{D})$, and $\Delta u = 0$ in D. Can $(\frac{\partial u}{\partial x_1})^2$ have an interior maximum in D? Justify your answer.

3.a.) (10 pts) Let Ω be a bounded C^2 domain in $\mathbf{R^n}$. Assume:

u, u_t, u_{x_i}, and $u_{x_i x_j}$ are in $C(\overline{\Omega} \times [0, \infty))$ for all $1 \leq i, j \leq n$,
$u_t - \Delta u = 0$ in $\Omega \times [0, \infty)$,
and $u = 0$ in $\partial\Omega \times [0, \infty)$.

Show that for each $T > 0$:

(*) $\int_\Omega u^2(x, T)dx \leq \int_\Omega u^2(x, 0)dx.$

Hint: $0 = 2u(u_t - \Delta u)$ in $\Omega \times (0, T)$. Integrate by parts.

3.b.)(10 pts.) By integrating by parts as in 3.a) on $B_R(0) \times (0, T)$ and letting $R \to \infty$, prove that if v is a continuous, bounded solution in $\mathbf{R^n} \times [\mathbf{0}, \infty)$ of:

$v_t - \Delta v = 0$ in $\mathbf{R^n} \times [\mathbf{0}, \infty)$,
$v(x, 0) = f(x)$ for all x in $\mathbf{R^n}$,
$\int_{\mathbf{R^n}} |f(x)|^2 dx < \infty$,
where f is C^∞ with compact support in $\mathbf{R^n}$,
and $v_t, v_{x_i}, v_{x_i x_j}$ are in $C(\mathbf{R^n} \times [\mathbf{0}, \infty))$ for all $1 \leq i, j \leq n$,

then $\int_{\mathbf{R^n}} |v(x, T)|^2 dx \leq \int_{\mathbf{R^n}} |v(x, 0)|^2 dx.$

4.) Consider the solution of
$u_{tt} - \Delta u = 0$ in $\mathbf{R^3} \times (\mathbf{0}, \infty)$,
$u(x,0) = 0$ and $u_t(x,0) = g(x)$ for all x in $\mathbf{R^3}$,

where u is in $C^2(\mathbf{R^3} \times [\mathbf{0}, \infty)$. Assume g is C^∞ with compact support in $\mathbf{R^3}$ and $g(x) > 0$ when $|x| < 1$, $g(x) = 0$ when $|x| \geq 1$.

a.) (10 pts.) What is the solution to the above problem?

b.) (10 pts.) For each x_0 in $\mathbf{R^3}$, identify $Z(x_0) \equiv \{t > 0 : u(x_0, t) = 0\}$. Justify your answer.

5. Let $\Gamma = \{(x,y) \in \mathbf{R^2} : y = x^2\}$. Consider the Cauchy problem:
$4yu_{yy} - 4xu_{xy} + 3xy^2 u_{xx} = 0$,
$u(x,y) = x^3y^2 - 2y$ on Γ,
$u_y(x,y) = 3xy^2$ on Γ.

(a.) (10 pts.) At what points (x_0, y_0) on Γ is there a real analytic solution of this problem in a neighborhood of (x_0, y_0)? Cite a theorem to justify your answer.

b.) (10 pts.) Compute the terms of order ≤ 1 in the power series for the solution expanded about the point $(1,1)$.