MA 523 Qualifying Exam, Spring 2011. Name: ______________________

There are five questions, each worth 20 points.

1. Let $I(x)$ be the indicator function for $B_1(0) = \{x \in \mathbb{R}^3 \colon |x| < 1\}$. Let u be such that

$$-\Delta u = I(x) \qquad \text{for } x \in \mathbb{R}^3,$$

$$u(x) \to 0 \qquad \text{as} \qquad |x| \to \infty.$$

   a) Represent u using the fundamental solution of Laplace's equation.

   b) Prove that there are constants $0 < A_1 < A_2 < \infty$ so that

$$\frac{A_1}{|x|} \le u(x) \le \frac{A_2}{|x|}$$

   for all $|x| \ge 2$.

2. Let $u(x,t)$ solve

$$\begin{cases} u_{tt} - u_{xx} = f(x) & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

where f is C^1,

$$\begin{cases} 0 < f(x) \leq 1 & \text{if } |x| < 1, \\ f = 0 & \text{if } |x| \geq 1. \end{cases}$$

a) Give a formula for u and verify that it is a solution to the problem.

b) Describe the region in $\mathbb{R} \times [0, \infty)$ where $u(x,t)$ vanishes. Justify your answer.

3. Let $u(x,t)$ solve

$$\begin{cases} u_t - \Delta u = 0 & \text{for } x \in \mathbb{R}^2,\ t > 0 \\ u(x,0) = I(x) & \text{for } x \in \mathbb{R}^2 \end{cases}$$

where $I(x)$ is the indicator function for $B_1(0) = \{x \in \mathbb{R}^2 \colon |x| < 1\}$

a) Represent u using the fundamental solution to the heat equation.

b) Evaluate $u(0,t)$ for $t > 0$. Find the limits $\lim_{t\to\infty} t^2 u_t(0,t)$ and $\lim_{t\downarrow 0} t^{-1} u_t(0,t)$.

4. Let Ω be a bounded domain in $\mathbb{R}^2$ with a smooth boundary. Let $u \in C^2(\overline{\Omega} \times [0, \infty))$ satisfy

$$\begin{cases} u_t - \Delta u = 0 & \text{for } x \in \Omega,\ t > 0, \\ \nu \cdot \nabla u = u_\nu(x,t) = 0 & \text{for } x \in \partial\Omega,\ t > 0, \end{cases}$$

where $\nu(x)$ is the exterior normal to $\partial\Omega$ at x. Prove that $E_1(t) = \displaystyle\int_\Omega u^2\, dx$ and

$$E_2(t) = \int_\Omega |\nabla u|^2\, dx$$

are such that $\frac{dE_1}{dt}(t) \leq 0$ and $\frac{dE_2}{dt}(t) \leq 0$.

5. Consider the Cauchy problem for $u = u(x, y)$,

$$\begin{cases} uu_{xx} + u_x u_{xy} + u_y u_{yy} = u^2 & \text{for } (x, y) \in \mathbb{R}^2, \\ u(x, 0) = x^2 - 1 & \text{for } x \in \mathbb{R}, \\ u_y(x, 0) = x & \text{for } x \in \mathbb{R}. \end{cases}$$

If possible, find $u_{xx}(P)$, $u_{xy}(P)$, and $u_{yy}(P)$. If it is not possible to find all of these derivatives state the condition that is violated.

Do this for the two cases:

a) $P = (0, 0)$ and b) $P = (1, 0)$.