**Name:**

# Math 523 Qualifying Exam August 2013

**Note:** Some of the subquestions (a), (b), etc., in problems 1 and 2 are independent of each other and can be answered even if the previous ones are not answered. You have to justify each answer on this exam.

**1. (25 pts)**

(a) Write down the fundamental solution of the Laplace operator $\Delta$ in $\mathbf{R}^3$ and explain why it is a fundamental solution (state what a fundamental solution is but do not prove anything).

(b) Given a continuous function $f(x)$ of compact support, write down a particular solution of the equation $\Delta u = f$ in $\mathbf{R}^3$ in integral form. Prove that this solution is bounded, and converges to 0, as $|x| \to \infty$. Is the solution unique in the class of the bounded functions on $\mathbf{R}^3$? Is it unique in the class of the functions converging to 0, as $|x| \to \infty$?

(c) Let $\Omega$ be a bounded domain in $\mathbf{R}^3$ with smooth boundary. Explain the notion of Green's function $G(x, \xi)$ in $\Omega$ and how it is related to the fundamental solution of $\Delta$.

(d) Show that for each fixed $\xi \in \Omega$,

$$\frac{\partial}{\partial \nu_x} G(x, \xi) \geq 0 \quad \text{for } x \in \partial\Omega,$$

where $\nu$ is the outer unit normal to $\partial\Omega$.

(e) For each fixed $\xi \in \Omega$, evaluate

$$\int_{\partial\Omega} \frac{\partial}{\partial \nu_x} G(x, \xi) \, dS_x.$$

**2. (18 pts)** The Schrödinger equation in $\mathbf{R}^n$ is given by

$$-iu_t = \Delta u, \quad x \in \mathbf{R}^n,\ t \in \mathbf{R}, \tag{1}$$

where $i = \sqrt{-1}$. Consider the Scrödinger equation (1) with initial condition

$$u|_{t=0} = f(x). \tag{2}$$

(a) Is the initial value problem (1), (2) a Cauchy problem?

(b) Let $u$ be a classical solution of (1), (2) with $f \in C_0^\infty$. Show that the energy

$$E(t) := \int_{\mathbf{R}^n} |u(x,t)|^2 \, dx$$

remains constant. Here, $|u|^2 = u\bar{u}$, and $u$ is complex-valued. Use the following fact: $u$ and its derivatives decay faster than $C(t)(1+|x|)^{-N}$ for every $N$ and $t$.

(c) If there is a solution to (1), (2), with a compactly supported $f$, is it unique (in the class of classical solutions)?

(d) Find an oscillatory solution to (1) of the form

$$u_\xi(x,t) = e^{i(\lambda t + x\cdot\xi)},$$

where $\xi \in \mathbf{R}^n$, with some $\lambda = \lambda(\xi)$.

(e) Using the result in (d), write a formula for the formal solution of (1), (2) in the form of an integral with respect to $\xi$, involving the explicit expression for $u_\xi$ that you found above and the Fourier transform $\hat{f}(\xi)$ of $f$. Assume that $|\hat{f}|$ is integrable. Do not simplify that formula.

**3. (15 pts)** For the equation

$$xu_x + 2yu_y + u_z = 3u,$$

(a) Find a general solution of the equation.

(b) Describe all planes $P = \{ax + by + cz = d\}$ through the point $(1, 1, 1)$ with the property that there is a guaranteed unique solution of the equation near that points with any prescribed $C^1$ data on $P$.

**4. (15 pts)** Consider the heat equation in a bounded domain $\Omega \subset \mathbf{R}^n$ with smooth boundary with Neumann boundary conditions

$$
\begin{aligned}
u_t &= \Delta u, \quad x \in \Omega,\ t > 0, \\
u|_{t=0} &= f(x) \quad \text{for } x \in \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{for } x \in \partial\Omega,\ t > 0.
\end{aligned}
$$

(a) If $u$ is a classical solution, prove that the "energy"

$$E(t) = \int_\Omega |u(x,t)|^2 \, dx$$

is non-increasing.

(b) Is the solution, if exists, unique?

**5. (15 pts)** The Klein-Gordon equation is given by

$$u_{tt} - \Delta u + m^2 u = 0, \quad x \in \mathbf{R},\ t \in \mathbf{R},$$

where $m > 0$ is fixed.

(a) Formulate the Cauchy problem for that equation with Cauchy data at $t = 0$. Is that problem solvable (near any point on the plane $t = 0$)?

(b) Formulate the Duhamel's principle for the problem

$$u_{tt} - \Delta u + m^2 u = h(x,t), \quad x \in \mathbf{R},\ t > 0, \tag{3}$$

with zero Cauchy data at $t = 0$ and write down the solution to (3) as an integral involving a solution of a certain homogeneous equation. Prove that the procedure that you found provides indeed a solution of (3) with zero Cauchy data at $t = 0$. Assume that all integrals that you will get are convergent and differentiation under the integral sign is justified.