Math 523
Qualifying Examination
January 6, 2013
Prof. N. Garofalo

Name..

I. D. no. ....

Problem	Score	Max. pts.
1		30
2		40
3		30
4		40
5		30
Total		170

Problem 1. Prove that the solution to the Cauchy problem

$$\begin{cases} \Delta u - u_{tt} = 0, \qquad \text{in } \mathbb{R}^3 \times (0,\infty), \\ |x|u(x,0) = \sin|x|, \quad u_t(x,0) = 0, \qquad x \in \mathbb{R}^3, \end{cases}$$

is

$$u(x,t) = \frac{\sin(|x|+t) + \sin(|x|-t)}{2|x|}.$$

Hint: Notice that, by uniqueness, u must be spherically symmetric in $x \in \mathbb{R}^3$. Consider the function $v = ru$, where $r = |x|$, and solve the Cauchy problem in the two variables r and t satisfied by v.

Problem 2. Let $A \subset \mathbb{R}^n$, $n \geq 3$, be a C^2, connected, bounded open set, and let $\Omega = \mathbb{R}^n \setminus \overline{A}$.
(i) Prove that if the problem

$$\begin{cases} \Delta u \ = \ 0 & \text{in} \quad \Omega \ , \\ u \ = \ 1 & \text{on} \quad \partial\Omega \ , \\ \lim_{|x|\to\infty} \ u(x) \ = \ 0 & \end{cases}$$

has a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$, it must be unique.
(ii) Show that if $A = B(0, R) = \{x \in \mathbb{R}^n \mid |x| < R\}$, then u must be spherically symmetric and find an explicit formula for the solution u.

Problem 3. Let $\phi \in C_0^\infty(\mathbb{R}^n)$, and let $\omega \in \mathbb{R}^n$ be given. Using Fourier transform find the solution of the Cauchy problem

$$\begin{cases} \Delta f + < \omega, \nabla f > - \frac{\partial f}{\partial t} = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ f(x, 0) = \phi(x), & x \in \mathbb{R}^n. \end{cases}$$

Problem 4. Let

$$G(x-y;t-s)=\begin{cases}(4\pi(t-s))^{-\frac{n}{2}}\ \exp\left(-\frac{|x-y|^2}{4(t-s)}\right)\ , & t>s\ ,\\ 0, & t\le s,\end{cases}$$

be the fundamental solution of the heat equation $Hu=\Delta u-u_t=0$ in R^{n+1}, with singularity at (y,s). Let $u\in C^{2,1}(\mathbb{R}^n\times[0,T])$ be a function satisfying, together with its derivatives of order up to two, the inequality

$$|u(x,t)|\le Ae^{a|x|^2},\qquad x\in\mathbb{R}^n,\ t\ge 0, \tag{1}$$

for some $A,a>0$. Consider the function

$$\phi(R)\overset{def}{=}\int_{\mathbb{R}^n}u^2(y,R^2)G(x-y;T-R^2)dy,\qquad 0\le R<\sqrt{T},$$

where $x\in\mathbb{R}^n$ is fixed and the number $T>0$ is arbitrarily chosen.

(i) Prove that

$$\begin{aligned}\phi'(R)=&-4R\int_{\mathbb{R}^n}u(y,R^2)Hu(y,R^2)G(x-y;T-R^2)dy\\&-4R\int_{\mathbb{R}^n}|\nabla u(y,R^2)|^2G(x-y;T-R^2)dy.\end{aligned}$$

(**Note:** You can take for granted that, thanks to (1), all differentiations under the integral sign are legitimate, and the relative integrals convergent. So, do not worry about this).

(ii) By letting $R\to\sqrt{T}$, and using the property $\lim\limits_{t\to 0^+}\int_{\mathbb{R}^n}G(x-y,t)\varphi(y)dy=\varphi(x)$, infer from (i) that if u satisfies the differential inequality

$$u(x,t)Hu(x,t)\ge 0\qquad (x,t)\in\mathbb{R}^n\times(0,\infty),$$

and $u(x,0)=0$ for $x\in\mathbb{R}^n$, then it must be $u\equiv 0$ in $\mathbb{R}^n\times(0,\infty)$.

Problem 5. Let $B = B(0, R) = \{x \in \mathbb{R}^3 \mid |x| < R\}$.
(i) Prove that the problem

$$\begin{cases} \Delta u = -\lambda u \qquad \text{in} \quad B, \quad \lambda > 0, \\ u = 0, \qquad \text{on} \quad \partial B, \end{cases}$$

has infinitely many solutions couples $\{(\lambda_k, u_k)\}_{k\in\mathbb{N}}$, with $\lambda_k \nearrow \infty$ and $u_k \in C^\infty(\overline{B})$.
(ii) Find an explicit formula for λ_k and u_k.