MA52300 Qualifying Examination

August 2014 — Professors P. Bauman, A. Petrosyan, D. Phillips

1. Consider the Cauchy problem:

$$xu_x + yu_y = 2u,$$
$$u(x, 1) = x^3.$$

(a) Use the method of characteristics to find a formula for a C^1 solution $u = u(x, y)$ defined in some neighborhood of the line $y = 1$ in $\mathbb{R}^2$. [12pt]

(b) Is the solution you found in part (a) unique among all C^1 solutions in some (possibly smaller) neighborhood of the line $y = 1$? Explain your reasoning. [8pt]

2. Reduce the second order equation to a canonical form and solve the Cauchy problem: [20pt]

$$4y^2u_{xx} + 2(1-y^2)u_{xy} - u_{yy} - \frac{2y}{1+y^2}(2u_x - u_y) = 0$$
$$u(x,y)|_{y=0} = \phi(x), \quad u_y(x,y)|_{y=0} = \psi(x)$$

3. Let $u \in C^2(\mathbb{R}^n)$ be harmonic in $\mathbb{R}^n$. Prove that u is constant in each of the following cases.

(a) There exists a constant C such that [10pt]

$$\int_{B(x,1)} |u(y)|dy \leq C \quad \text{for any } x \in \mathbb{R}^n.$$

[*Hint:* First show that u is bounded in $\mathbb{R}^n$.]

(b) u is radial, i.e., $u(x) = \phi(|x|)$ for some function ϕ. [10pt]

4. Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary and $u \in C^2(\overline{\Omega} \times [0, \infty))$ a solution of the following problem

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \Omega \times (0, \infty) \\ u &= 0 \quad \text{on } \partial\Omega \times (0, \infty) \\ u = g &\geq 0 \quad \text{on } \Omega \times \{0\}. \end{aligned}$$

(a) Show that $u(x, t) \geq 0$ in $\Omega \times (0, \infty)$ and $\partial u / \partial \nu \leq 0$ on $\partial\Omega \times (0, \infty)$, where ν is the outer normal on $\partial\Omega$. [10pt]

(b) Let $E(t) := \int_\Omega u^2(x, t)\, dx$. Show that $E(t)$ is a nonincreasing function of t. [10pt]

5. Let $u(x,t)$ solve

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)$$
$$u(x,0) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$
$$u_t(x,0) = \begin{cases} 2 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

(a) Give representations for u for the cases $n = 2$ and $n = 3$. Work out explicit formulas for $u(0,t)$ for $t \ge 0$ in each case. [12pt]

(b) State Huygen's principle and explain the connection between the principle and the two examples. [8pt]