1. Let u be a harmonic function in a connected open set $\Omega \subset \mathbb{R}^n$. It is known that $|\nabla u| \leq 1$ in Ω and $|\nabla u(x_0)| = 1$ for some $x_0 \in \Omega$. Prove that there exists a unit vector $e \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $u(x) = e \cdot x + c$ for any $x \in \Omega$. [20pt]

[Hint. Consider the partial derivative $\partial_e u = e \cdot \nabla u$ in the direction $e = \nabla u(x_0)$.]

2. Consider the Cauchy problem [20pt]

$$u_{x_1}^2 + u_{x_2}^2 = u \quad \text{in } \mathbb{R}^2, \quad u(x_1, 0) = a x_1^2 \quad \text{for } x_1 \in \mathbb{R}.$$

(a) For which values of the positive constant a is there a (classical) solution? Is it unique?

(b) Find all solutions of the Cauchy problem for $a = 1/8$.

3. Let $u \in C^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$, $T > 0$ be such that [20pt]

$$\Delta u - u_t \leq 0 \quad \text{in } \mathbb{R}^n \times (0,T], \qquad u(\cdot, 0) \geq 0 \quad \text{on } \mathbb{R}^n$$

and

$$\liminf_{|x| \to \infty} u(x,t) \geq 0 \quad \text{uniformly in } t \in [0,T].$$

Prove that $u(x,t) \geq 0$ in $\mathbb{R}^n \times [0,T]$.

4. Let [20pt]

$$K_y(x) := \frac{2y}{\alpha_{n+1}(|x|^2 + y^2)^{(n+1)/2}} \quad \text{for} \quad x \in \mathbb{R}^n,\ y > 0,$$

where α_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$. Evaluate the convolution

$$(K_a * K_b)(x) = \int_{\mathbb{R}^n} K_a(z) K_b(x - z) dz$$

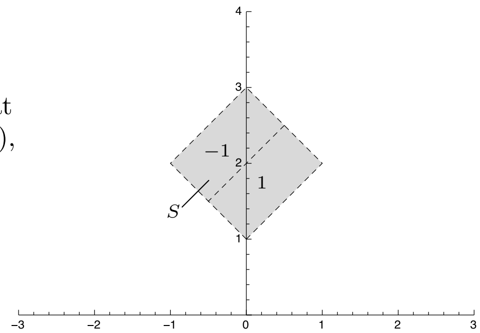
for $a > 0$, $b > 0$.

[Hint. Consider the Dirichlet problem for the Laplacian in $\mathbb{R}^{n+1}_+ = \{(x, y) : x \in \mathbb{R}^n, y > 0\}$ for a conveniently chosen initial data. Recall that $K_y(x)$ is the Poisson kernel for $\mathbb{R}^{n+1}_+$.]

5. Let u solve [20pt]

$$u_{tt} - u_{xx} = f \quad \text{in } \mathbb{R} \times (0, \infty)$$
$$u = 0,\ u_t = 0 \quad \text{on } \mathbb{R} \times \{t = 0\},$$

with f supported in a bounded set $S \subset \mathbb{R} \times (0, \infty)$. Suppose that S is a square lying in $\mathbb{R} \times (0, \infty)$ with corners at the points $(0, 1)$, $(1, 2)$, $(0, 3)$, $(-1, 2)$ and



$$f(x,t) = \begin{cases} -1, & \text{for } (x,t) \in S \cap \{t > x + 2\} \\ 1, & \text{for } (x,t) \in S \cap \{t < x + 2\} \\ 0, & \text{otherwise.} \end{cases}$$

Describe the shape of u for times $t > 3$.
[Hint. How many traveling waves are produced?]