MA 52300 QUALIFYING EXAMINATION
August 2019 (Professors Bauman and Phillips)

Instructions: This exam contains five problems. Show your reasoning in all problems.

1. (20 pt.) Assume f is a continuous real-valued function on $\mathbb{R}$ and f is uniformly bounded on $\mathbb{R}$, i.e.

$$|f(x)| \leq M \text{ for all } x \in \mathbb{R}$$

where $M > 0$. Assume $u(x,t)$ is a bounded continuous function on $\mathbb{R} \times [0,\infty)$ with u_t, u_x, u_{xx} continuous on $\mathbb{R} \times (0,\infty)$, satisfying

$$\begin{aligned} u_t - u_{xx} &= 0 \qquad \text{in } \mathbb{R} \times (0,\infty), \\ u(x,t) &= f(x) \quad \text{ for } x \in \mathbb{R}. \end{aligned}$$

a) State a formula for $u(x,t)$ in terms of an integral over $\mathbb{R}$.

b) Prove that if $f(x) \to 0$ as $|x| \to \infty$, then for each fixed $t > 0$, $\lim_{|x|\to\infty} u(x,t) = 0$ uniformly for $x \in \mathbb{R}$.

Problem 1 more space

2. (20 pt.) Given that $\frac{1}{2\pi} \ln |x|$ for $x = (x_1, x_2)$ is the fundamental solution for the Laplacian in $\mathbb{R}^2$, find the Green's function $G(x, y)$ for the region $D = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$.

Problem 2 more space

3. (20 pt.) Let Ω be an open connected set in $\mathbb{R}^n$ for $n \geq 2$. Assume $u \in C^2(\Omega)$ and $\Delta u \geq 0$ in Ω.

a) For any open ball $B_R(x_0)$ in $\mathbb{R}^n$ such that $\overline{B_R(x_0)} \subset \Omega$, prove that

$$u(x_0) \leq \fint_{\partial B_R(x_0)} u \, dS$$

where dS denotes surface measure.
(Hint: Use the divergence theorem and $0 \leq \int_{B_r(x_0)} (\Delta u) \, dx$ for all $0 < r \leq R$.)

b) Assume that $u \leq 0$ on Ω. Prove that if $u(\overline{x}) = 0$ for some $\overline{x}$ in Ω, then $u \equiv 0$ in Ω.

Problem 3 more space.

4. (20 pt.) Let $\mathbb{R}^2_+ = \{(x,t) : x \in \mathbb{R}, t > 0\}$. Assume $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Assume that $f = f(x,t)$ and f, f_x, f_{xx}, and f_t are continuous on $\overline{\mathbb{R}^2_+}$. Let $u(x,t) \in C^2(\overline{\mathbb{R}^2_+})$ satisfy:

$$\begin{array}{lcl} (i) & u_{tt} - u_{xx} = f(x,t) & \text{for all } (x,t) \in \mathbb{R}^2_+, \\ (ii) & u(x,0) = g(x),\ u_t(x,0) = h(x) & \text{for all } x \in \mathbb{R}. \end{array}$$

a) State a formula for $u(x,t)$ in terms of g, h, and f.

b) Let $h(x) = g(x) = 0$ for all $x \in \mathbb{R}$. Let $f(x,t)$ be the function defined on $\overline{\mathbb{R}^2_+}$ such that $f(x,t) = 1$ on the rectangle

$$R = \{(x,t) : -1 \leq x \leq 1,\ 0 \leq t \leq 1\}$$

and $f = 0$ at all other points in $\overline{\mathbb{R}^2_+}$. It is known that in this case, the formula for the solution $u(x,t)$ of problem (i) and (ii) is still valid. (It gives a weak solution that is continuous on $\overline{\mathbb{R}^2_+}$.)

Find all values of x (if any) such that $u(x,10) = 0$; also find all values of x (if any) such that $u(x,10) = 1$. Use the formula for $u(x,t)$ to give a rigorous proof that your answer is correct.

Problem 4 more space.

5. (20 pt.) Consider the solution $u(x, y)$ of:

$$u_x + uu_y = 1$$

defined for all (x, y) in some open set in $\mathbb{R}^2$ containing the half-line $L = \{(x, 0) : x > 0\}$ such that

$$u(x, 0) = e^x \text{ for } x > 0.$$

a) Find the integral curves $\big(x(s,t), y(s,t), z(s,t)\big)$ satisfying

$$x(0,t) = t, \;\; y(0,t) = 0, \;\; z(0,t) = e^t$$

corresponding to the solution.

b) Compute u_{yy} on L.

Problem 5 more space